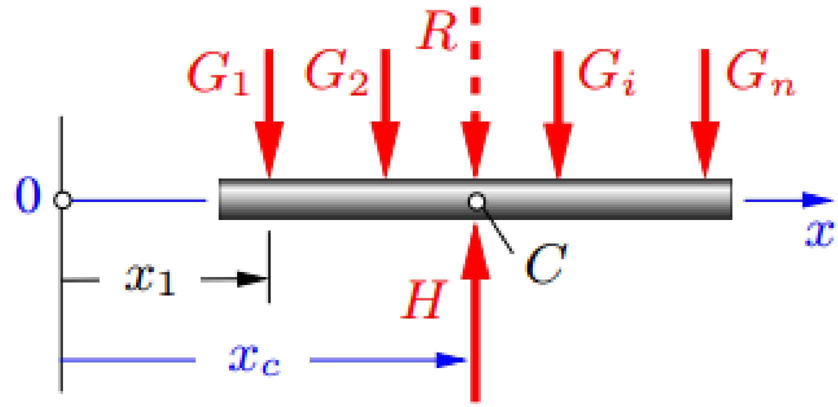
CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

1 Center of Forces

Applying the equilibrium conditions



$$\uparrow: \quad H - \sum G_i = 0; \qquad \overset{\curvearrowleft}{0}: \quad x_c H - \sum x_i G_i = 0$$

$$\boxed{x_c = \frac{\sum x_i G_i}{\sum G_i}}$$

The corresponding point C (an arbitrary point on the action line of H) is called the *center of forces*.

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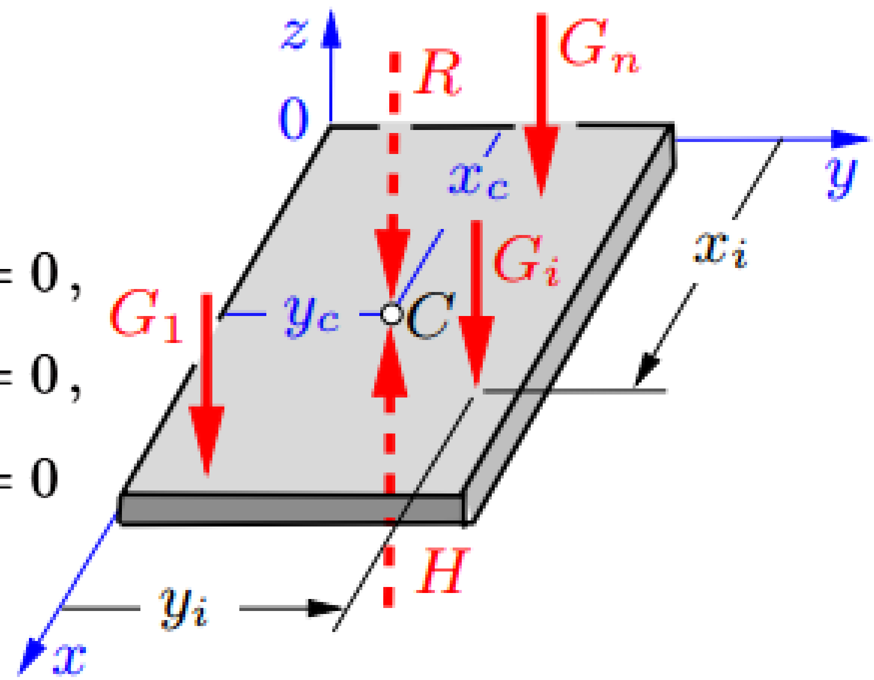
1 Center of Forces

Applying the equilibrium conditions



$$\sum F_{iz} = 0 : \qquad H - \sum G_i = 0 ,$$

$$\sum M_{ix}^{(0)} = 0 : \qquad y_c H - \sum y_i G_i = 0 ,$$

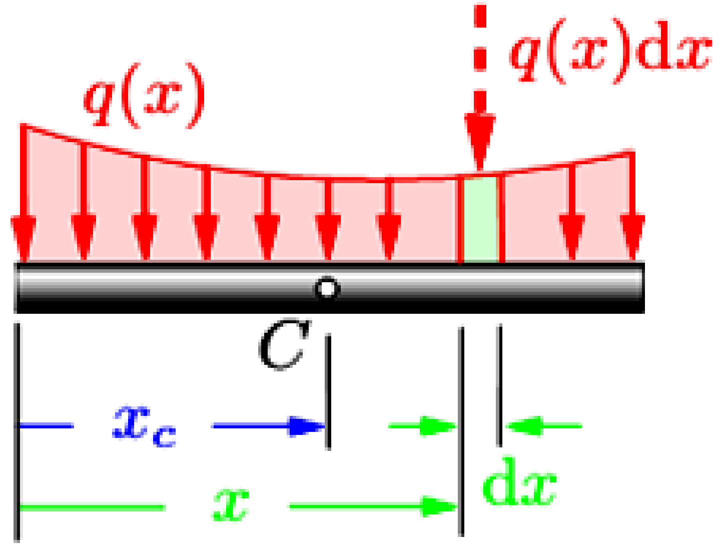
$$\sum M_{iy}^{(0)} = 0 : \qquad -x_c H + \sum x_i G_i = 0$$

$$x_c = \frac{\sum x_i G_i}{\sum G_i} , \quad y_c = \frac{\sum y_i G_i}{\sum G_i} .$$

CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

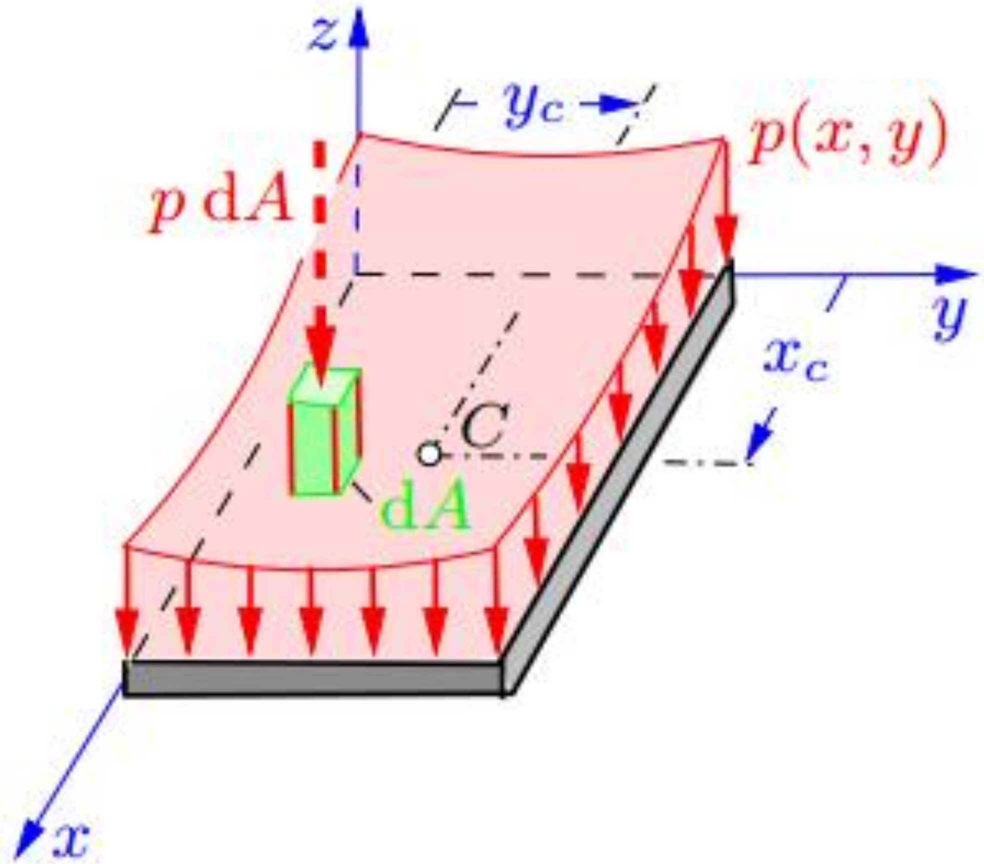
1 Center of Forces

$$x_c = \frac{\int x\, q(x)\, \mathrm{d}x}{\int q(x)\, \mathrm{d}x}\,.$$



CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

1 Center of Forces



$$x_c = \frac{\int x\,p(x,y)\,\mathrm{d}A}{\int p(x,y)\,\mathrm{d}A}\,, \qquad y_c = \frac{\int y\,p(x,y)\,\mathrm{d}A}{\int p(x,y)\,\mathrm{d}A}\,.$$

CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

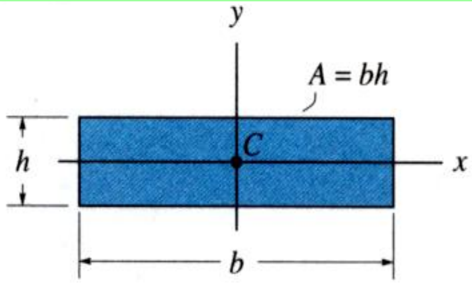
2 Center of Gravity and Center of Mass

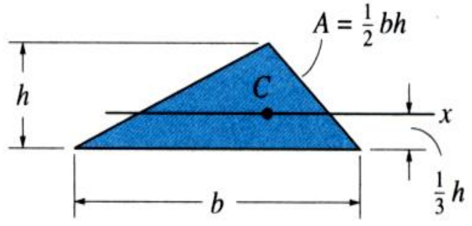
The center of gravity (G) is a point which locates the resultant weight of a system of particles or body.

The center of mass is a point which locates the resultant mass of a system of particles or body. Generally, its location is the same as that of G.

The centroid C is a point which defines the geometric center of an object.

The centroid coincides with the center of mass or the center of gravity only if the material of the body is homogenous (density or specific weight is constant throughout the body).





CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

2 Center of Gravity and Center of Mass

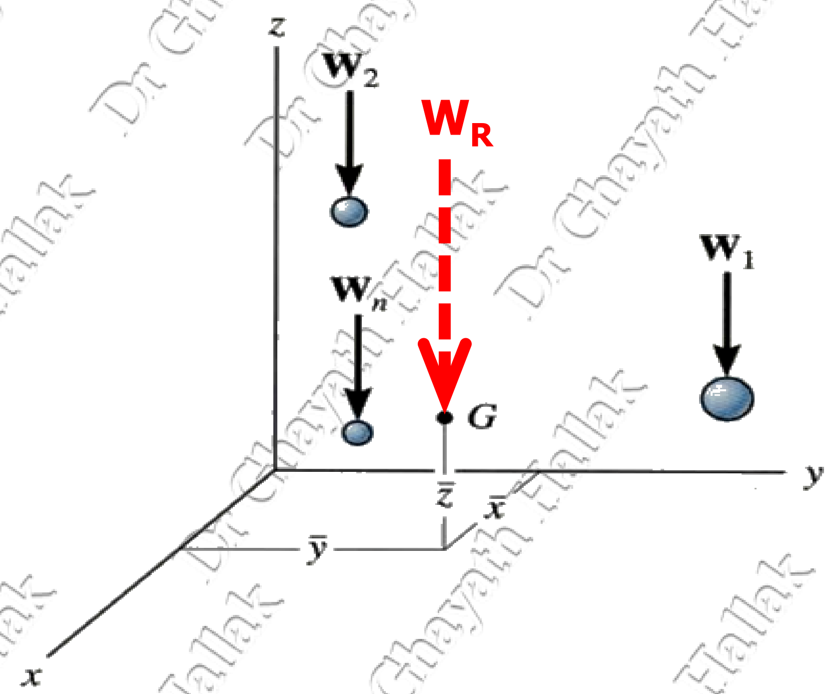
Consider a system of n particles as shown in the figure. The net or the resultant weight is given as $W_R = \Sigma W$.

Summing the moments about the y-axis, we get

$$\bar{x} W_R = x_1 W_1 + x_2 W_2 + \cdots x_n W_n$$

where x_1 represents x coordinate of W_1, etc..

Similarly, we can sum moments about the x- and z-axes to find the coordinates of G.



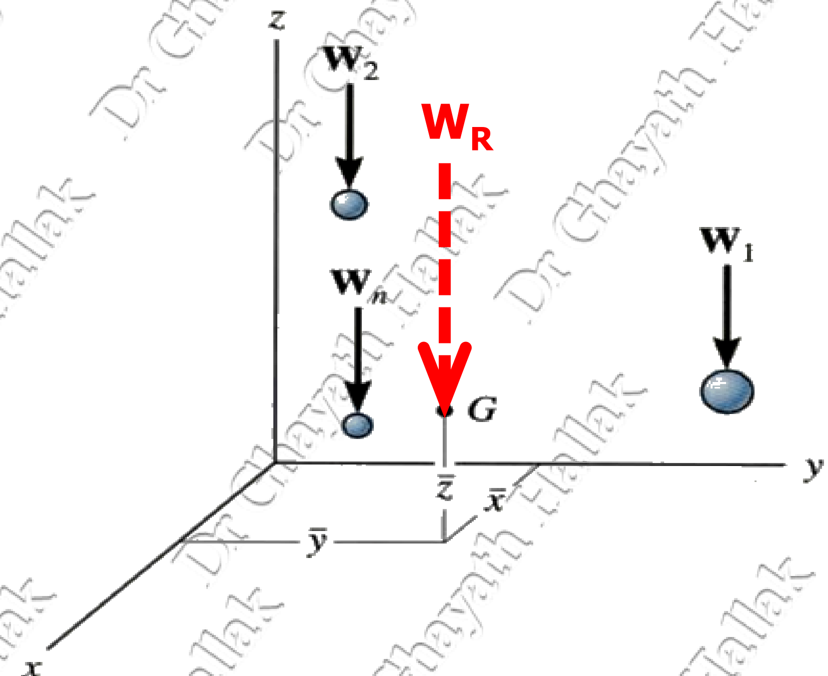
CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

2 Center of Gravity and Center of Mass

coordinates of the center of gravity

$$\bar{x} = \frac{\sum \tilde{x} W}{\sum W}; \bar{y} = \frac{\sum \tilde{y} W}{\sum W},$$

$$\bar{z} = \frac{\sum \tilde{z} W}{\sum W}$$

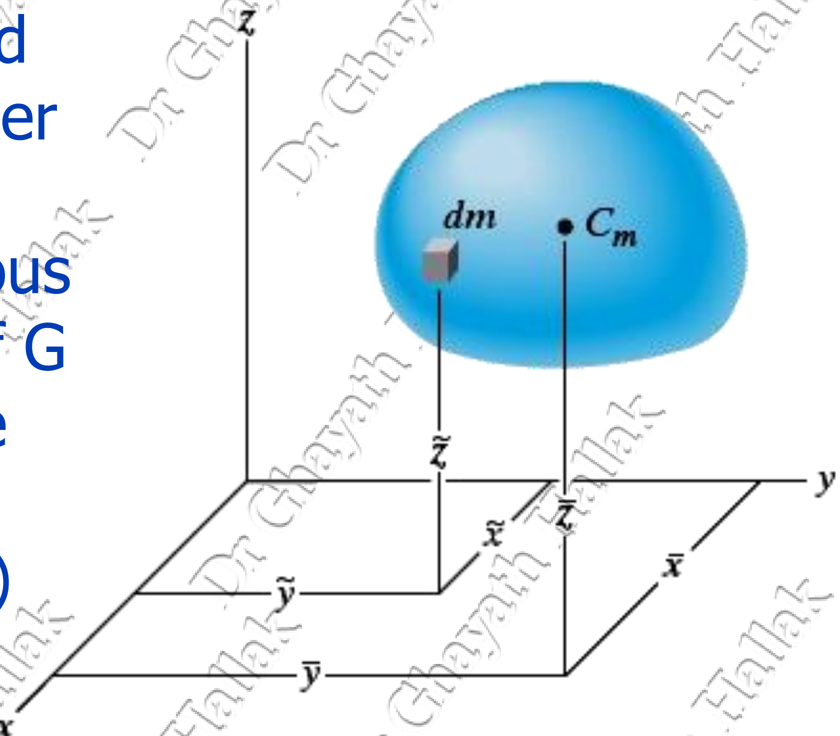


By replacing the W with a M in these equations, the coordinates of the center of mass can be found.

CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

2 Center of Gravity and Center of Mass

A **rigid body** can be considered as made up of an infinite number of particles. Hence, using the same principles as in the previous slide, we get the coordinates of G by simply replacing the discrete summation sign (Σ) by the continuous summation sign (∫) and W by dW.



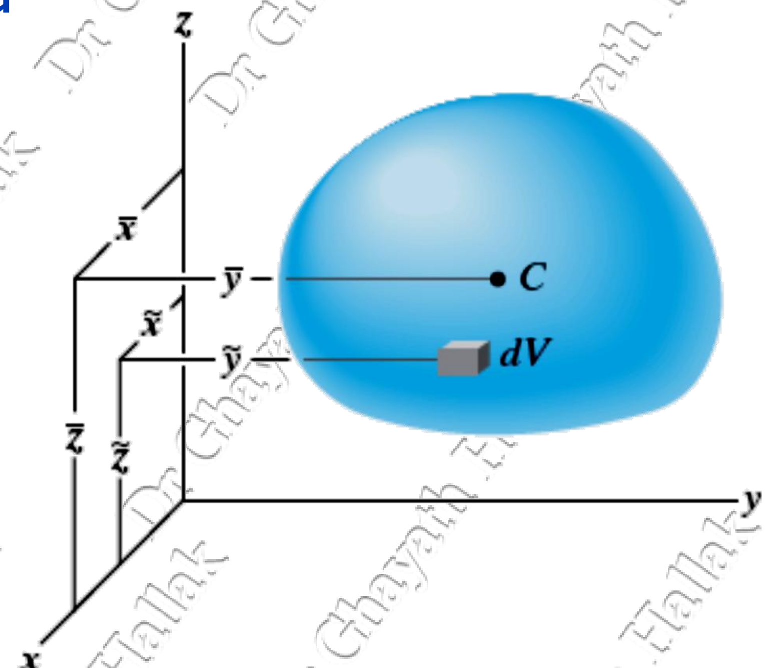
$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW};\ \bar{y} = \frac{\int \tilde{y}\,dW}{\int dW};\ \bar{z} = \frac{\int \tilde{z}\,dW}{\int dW}$$

CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

3 Center of Volume

- Consider an object subdivided into volume elements dV, for location of the centroid,

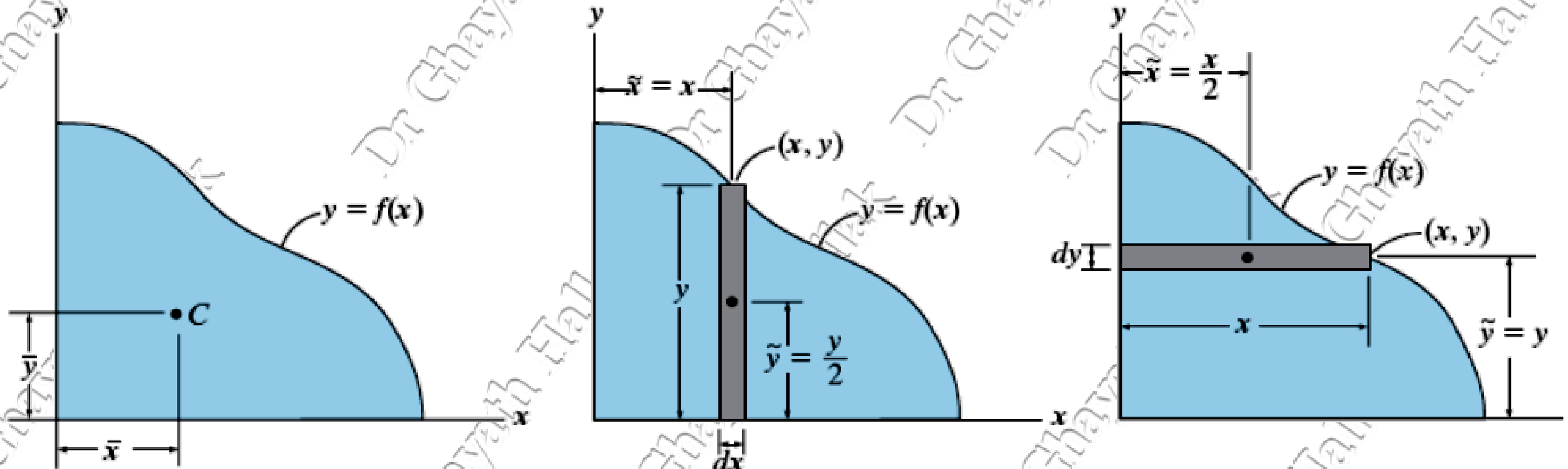
$$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV};\ \bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV};\ \bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV}$$



Similarly, the coordinates of the center of mass and the centroid of area, or length can be obtained by replacing V by m, A, or L, respectively.

CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

4 Centroid of an Area

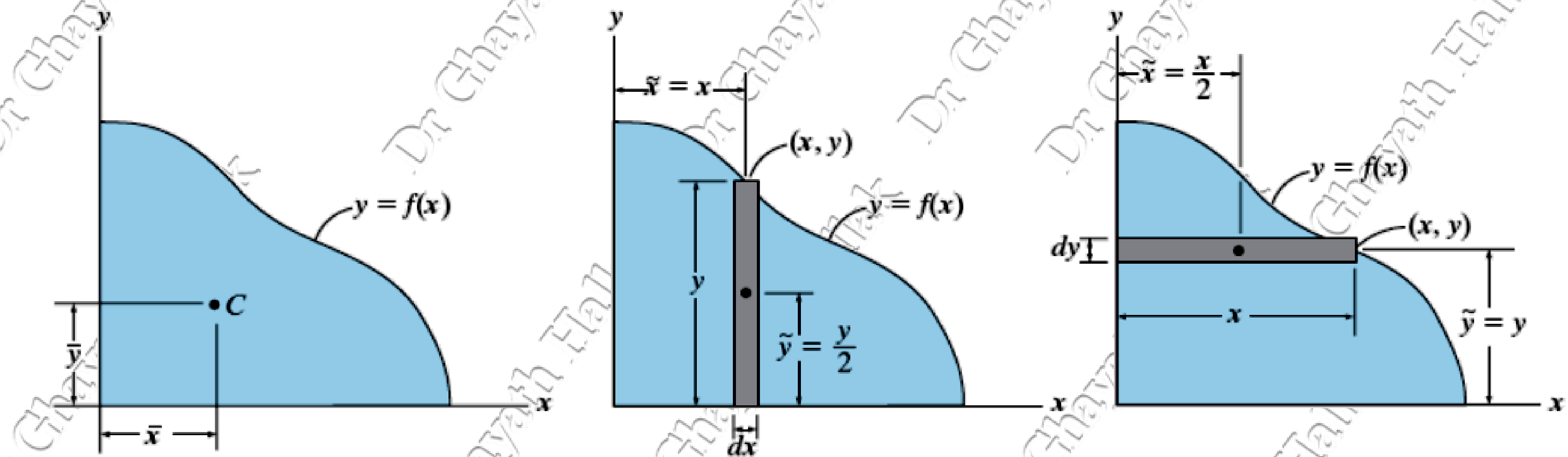


For centroid for surface area of an object, such as plate and shell, subdivide the area into differential elements dA

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA}\,;\ \bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA}$$

CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

4 Centroid of an Area



$$Q_x = \int_A \tilde{y} \; dA$$

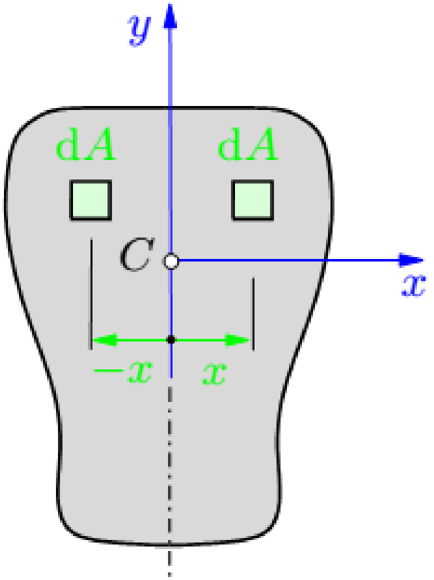
$$Q_y = \int_A \tilde{x} \; dA$$

The integrals are called the first moments of the area with respect to the x and the y-axis, respectively:

The first moments of an area with respect to axes through its centroid are zero.

CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

4 Centroid of an Area



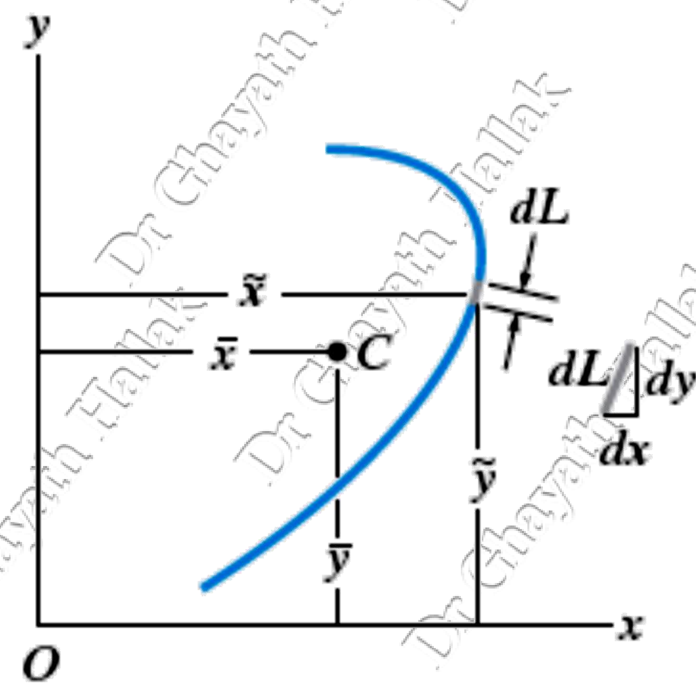
If the area has an axis of symmetry, the centroid of the area lies on this axis.

CENTER OF GRAVITY, CENTER OF MASS AND CENTROID FOR A BODY

4 Centroid of a Line

- If the geometry of the object takes the form of a line, the balance of moments of differential elements dL about each of the coordinate system yields

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL};\ \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL};\ \bar{z} = \frac{\int_L \tilde{z}\, dL}{\int_L dL}$$

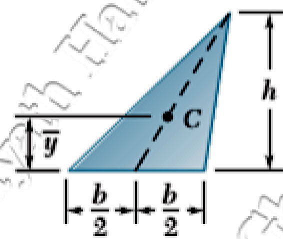
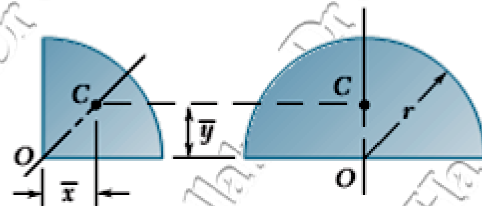
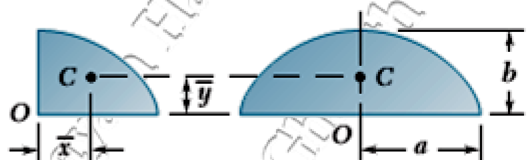
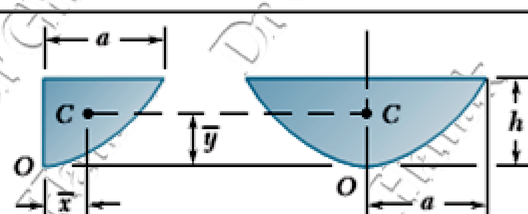
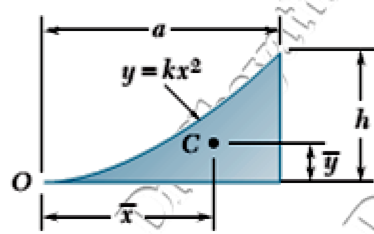


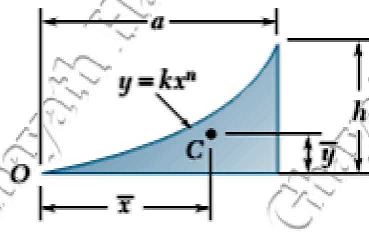
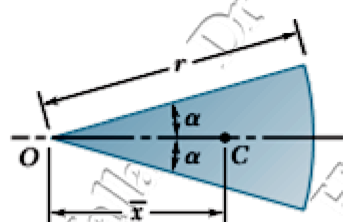
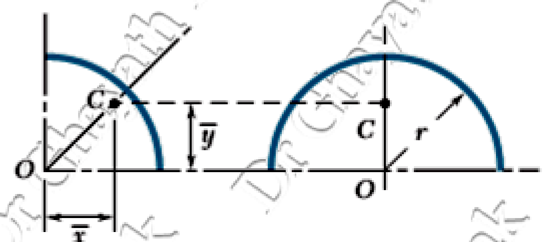
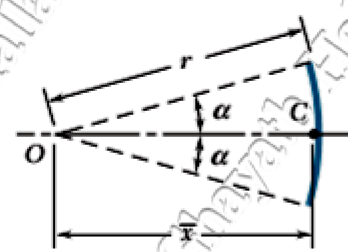
STEPS FOR DETERMING AREA CENTROID

1. Choose an appropriate differential element dA at a general point (x,y). Hint: Generally, if y is easily expressed in terms of x (e.g., $y = x^2 + 1$), use a vertical rectangular element. If the converse is true, then use a horizontal rectangular element.
2. Express dA in terms of the differentiating element dx (or dy).
3. Determine coordinates ($\tilde{x}$, $\tilde{y}$) of the centroid of the rectangular element in terms of the general point (x,y).
4. Express all the variables and integral limits in the formula using either x or y depending on whether the differential element is in terms of dx or dy, respectively, and integrate.

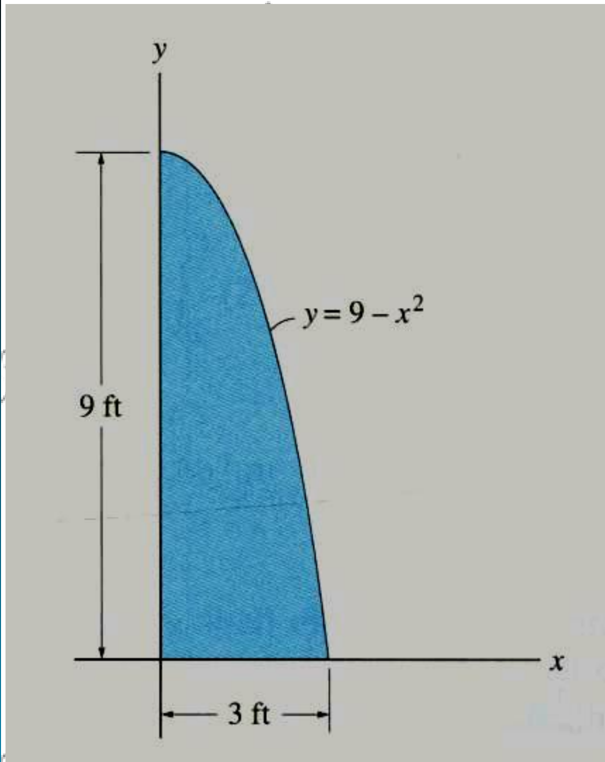
Note: Similar steps are used for determining CG, CM, etc..

These steps will become clearer by doing a few examples.

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

Shape		$\bar{x}$	$\bar{y}$	Area
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$

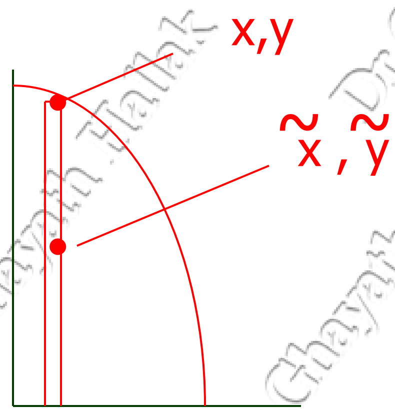
EXAMPLE



Given: The area as shown.

Find: The centroid location (x , y)

Plan: Follow the steps.



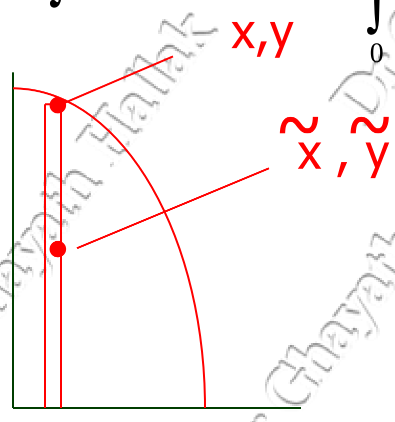
Solution

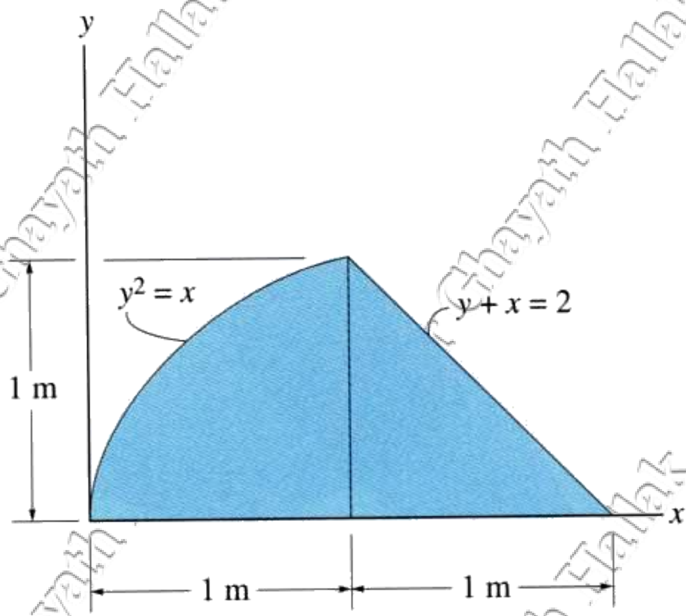
1. Since y is given in terms of x, choose dA as a vertical rectangular strip.
2. dA = y dx = $(9 - x^2)$ dx
3. $\tilde{x}$ = x and $\tilde{y}$ = y / 2

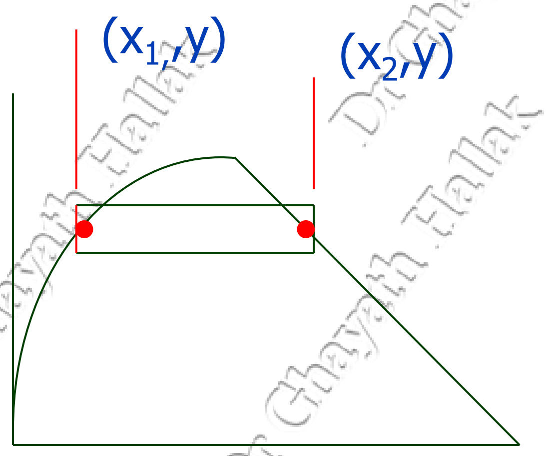
EXAMPLE

$$\bar{x} = \frac{\int \tilde{x}\,dA}{\int dA} = \frac{\int_0^3 x\left(9 - x^2\right)dx}{\int_0^3 \left(9 - x^2\right)dx} = \frac{\left[9\frac{x^2}{2} - \frac{x^4}{4}\right]_0^3}{\left[9x - \frac{x^3}{3}\right]_0^3} = \frac{20.25}{18} = 1.125$$

$$\bar{y} = \frac{\int \tilde{y}\,dA}{\int dA} = 0.5\frac{\int_0^3 \left(9 - x^2\right)\left(9 - x^2\right)dx}{\int_0^3 \left(9 - x^2\right)dx} = 0.5\frac{\left[81x - \frac{18x^3}{3} + \frac{x^5}{5}\right]_0^3}{\left[9x - \frac{x^3}{3}\right]_0^3} = 0.5\frac{129.6}{18} = 3.6$$







EXAMPLE

Given: The area as shown.

Find: The x of the centroid.

Plan: Follow the steps.

Solution

1. Choose dA as a horizontal rectangular strip.

2. $dA = (x_2 - x_1) \, dy$

   $= ((2 - y) - y^2) \, dy$

3. $\tilde{x} = (x_1 + x_2) / 2$

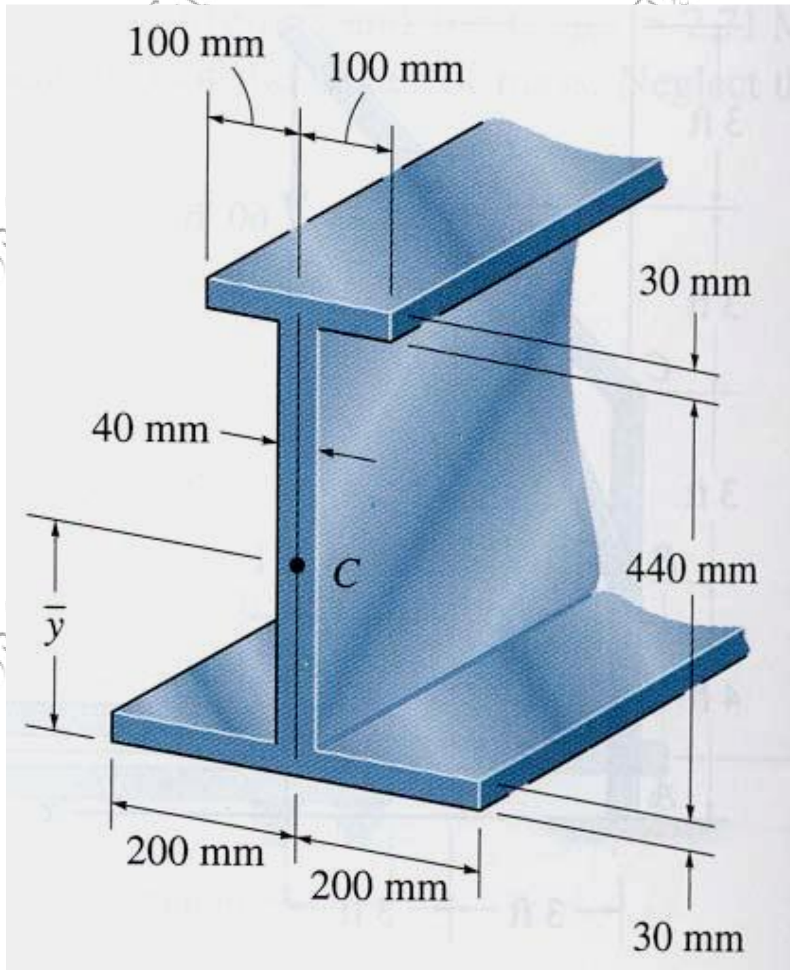
   $= 0.5 ((2 - y) + y^2)$

$$\bar{x} = \frac{\int \tilde{x}\,dA}{\int dA} = \frac{\int_0^1 0.5\left(2 - y + y^2\right)\left(2 - y - y^2\right)dy}{\int_0^1 \left(2 - y - y^2\right)dy} = \frac{\int_0^1 0.5\left(4 - 4y + y^2 - y^4\right)dy}{\int_0^1 \left(2 - y - y^2\right)dy}$$

$$= 0.5\frac{\left[4y - 4\frac{y^2}{2} + \frac{y^3}{3} - \frac{y^5}{5}\right]_0^1}{\left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1} = 0.5\frac{2.13}{1.167} = 0.91$$

$$\bar{y} = \frac{\int \tilde{y}\,dA}{\int dA} = \frac{\int_0^1 y\left(2 - y - y^2\right)dy}{\int_0^1 \left(2 - y - y^2\right)dy} = \frac{\left[2\frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1}{\left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1} = \frac{0.416}{1.167} = 0.357$$

COMPOSITE BODIES

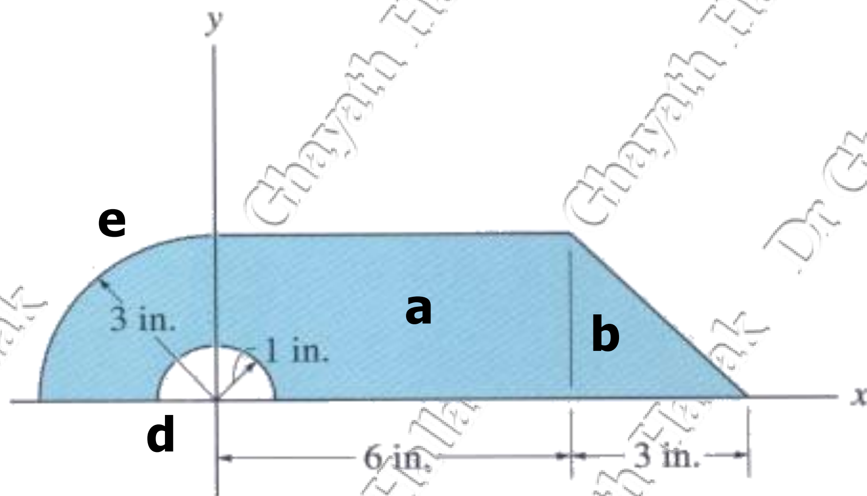


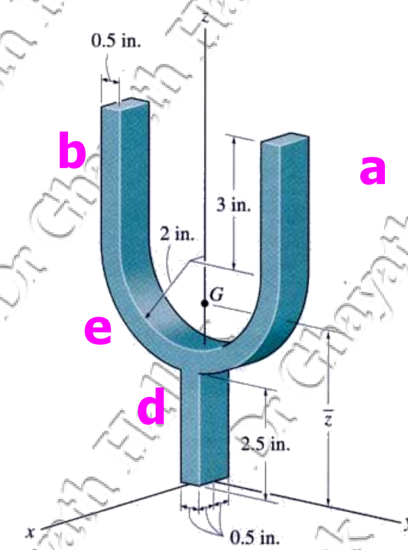
The I-beam is commonly used in building structures.

When doing a stress analysis on an I - beam, the location of the centroid is very important.

How can we easily determine the location of the centroid for a given beam shape?

CONCEPT OF A COMPOSITE BODY

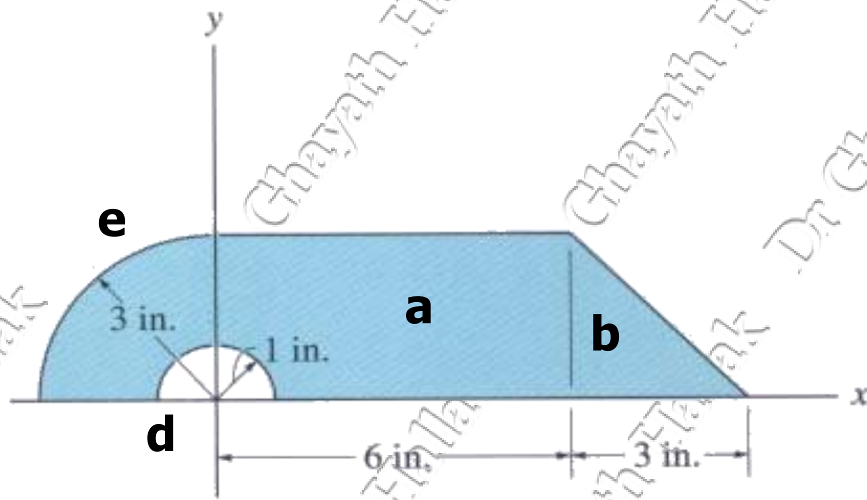


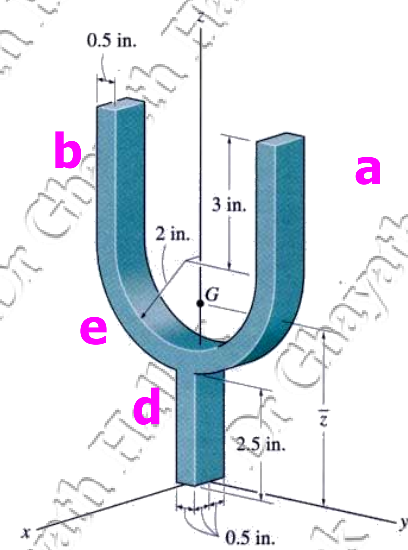


Many industrial objects & construction members can be considered as composite bodies **<u>made up of a series of connected "simpler" shaped parts or holes</u>**, like a rectangle, triangle, and semicircle.

Knowing the location of the centroid, C, or center of gravity, G, of <u>the simpler shaped parts</u>, we can easily determine the location of the C or G for the more complex composite body.

CONCEPT OF A COMPOSITE BODY





This can be done by considering each part as a "particle" and following the procedure as described earlier.

This is a **simple, effective, and practical method** **of** determining the location of the centroid or center of gravity for composite body or surface.

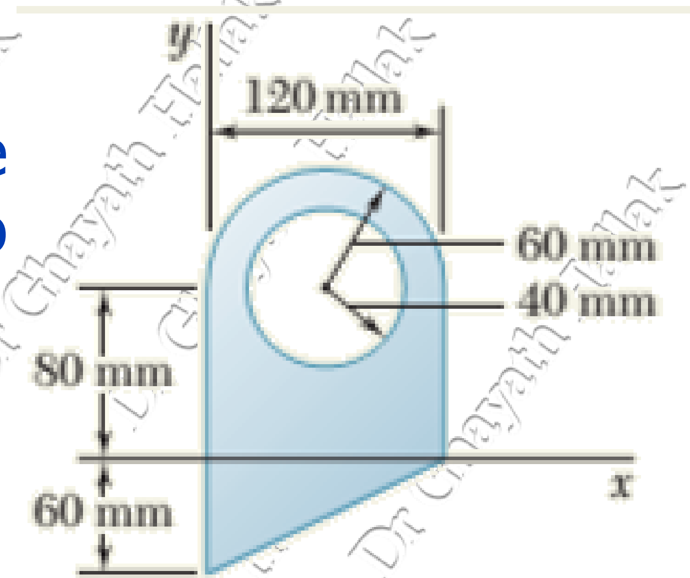
STEPS FOR ANALYSIS

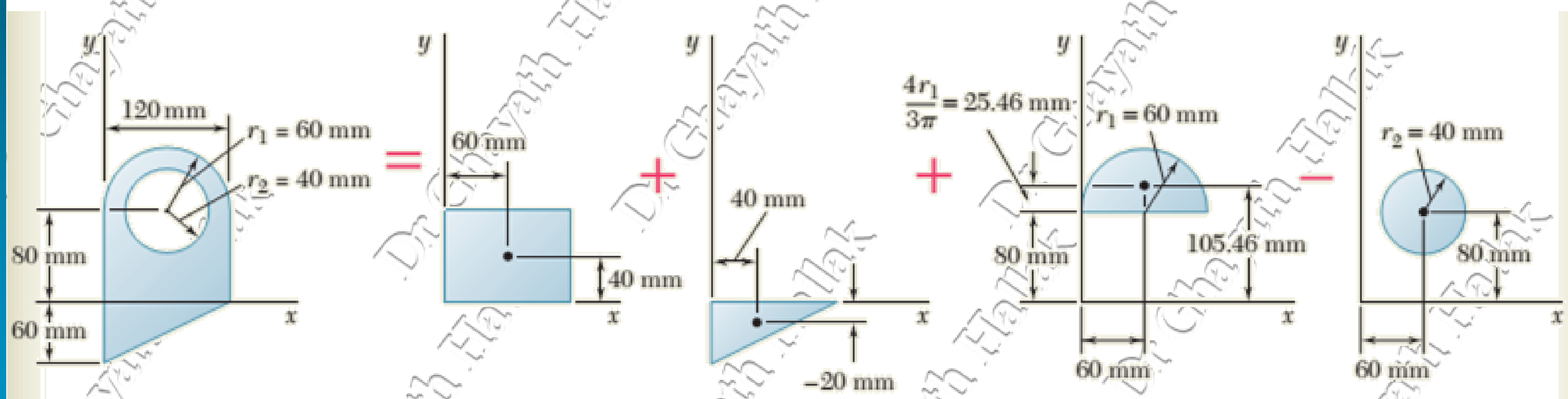
1. Divide the body into pieces that are known shapes. <u>Holes are considered as pieces with negative weight or size.</u>
2. Make a table with the first column for segment number, the second column for weight, mass, or size (depending on the problem), the next set of columns for the moment arms, and, finally, several columns for recording results of simple intermediate calculations.
3. Fix the coordinate axes, determine the coordinates of the center of gravity of centroid of each piece, and then fill-in the table.
4. Sum the columns to get x, y, and z. Use formulas like
5. $X_c = (\Sigma \tilde{x}_i A_i) / (\Sigma A_i)$ or $X_c = (\Sigma \tilde{x}_i W_i) / (\Sigma W_i)$

This approach will become clear by doing examples!

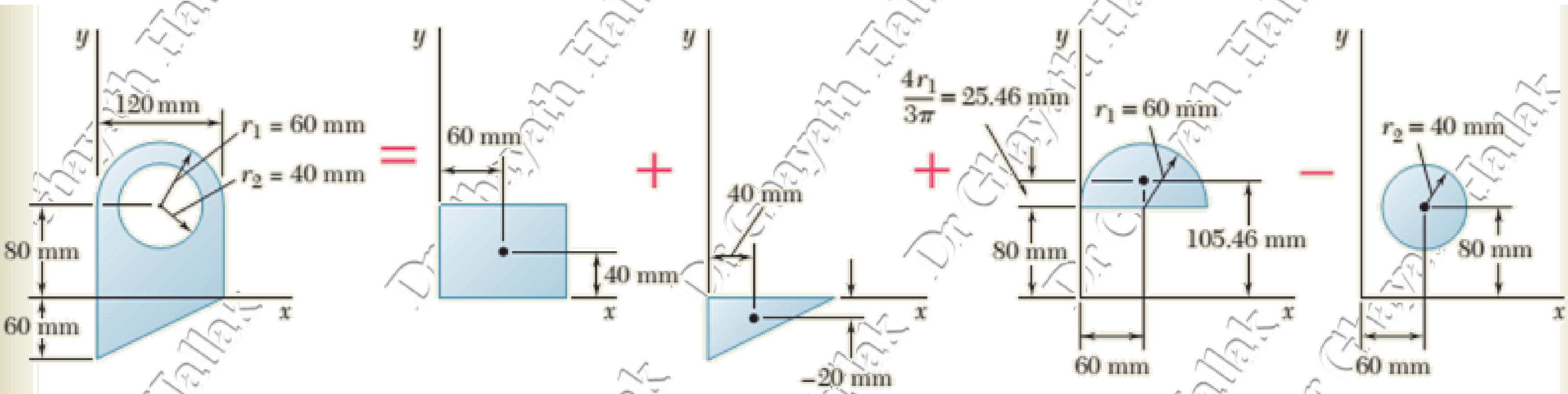
EXAMPLE

For the plane area shown, determine (*a*) the first moments with respect to the *x* and *y* axes, (*b*) the location of the centroid.





EXAMPLE



Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

a. First Moments of the Area. $Q_x = \Sigma\bar{y}A = 506.2 \times 10^3 \text{ mm}^3$

$$Q_y = \Sigma\bar{x}A = 757.7 \times 10^3 \text{ mm}^3$$

b. Location of Centroid.

$\bar{X}\Sigma A = \Sigma\bar{x}A$: $\quad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$

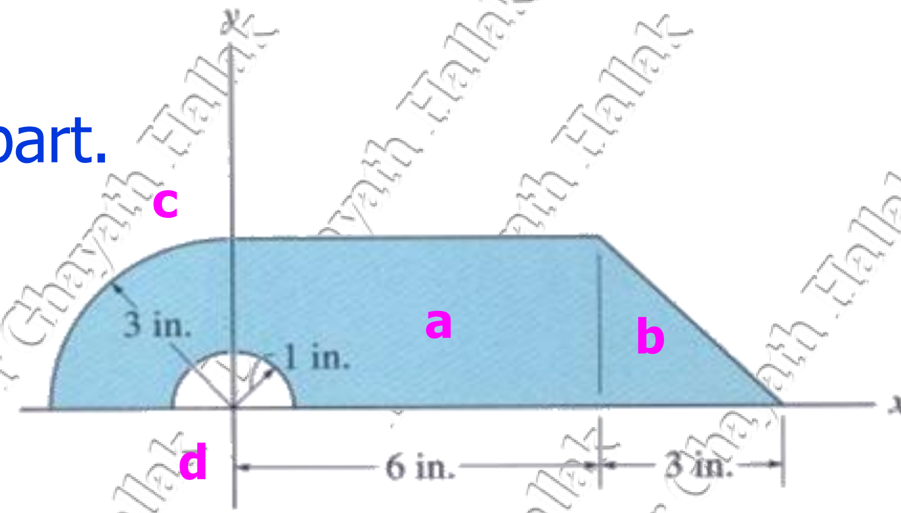
$$\bar{X} = 54.8 \text{ mm}$$

$\bar{Y}\Sigma A = \Sigma\bar{y}A$: $\quad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$

$$\bar{Y} = 36.6 \text{ mm}$$

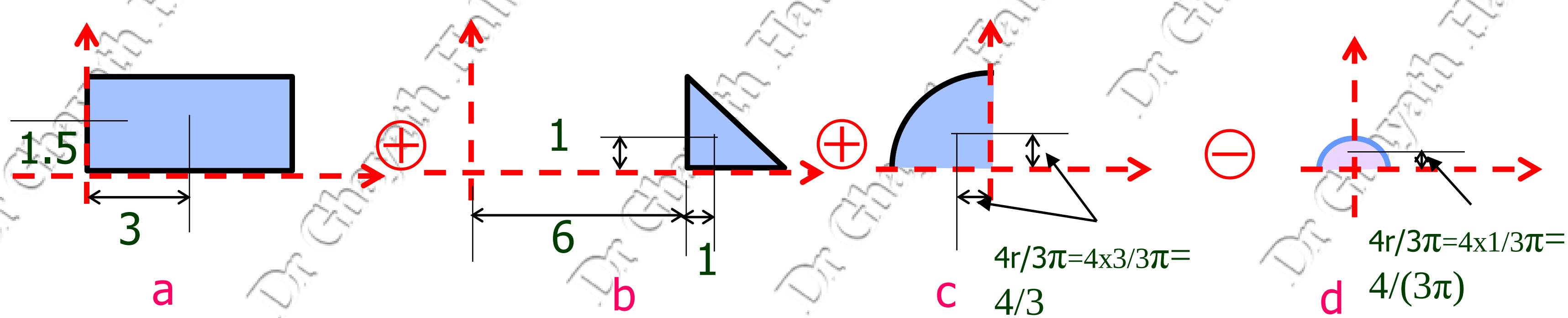
EXAMPLE

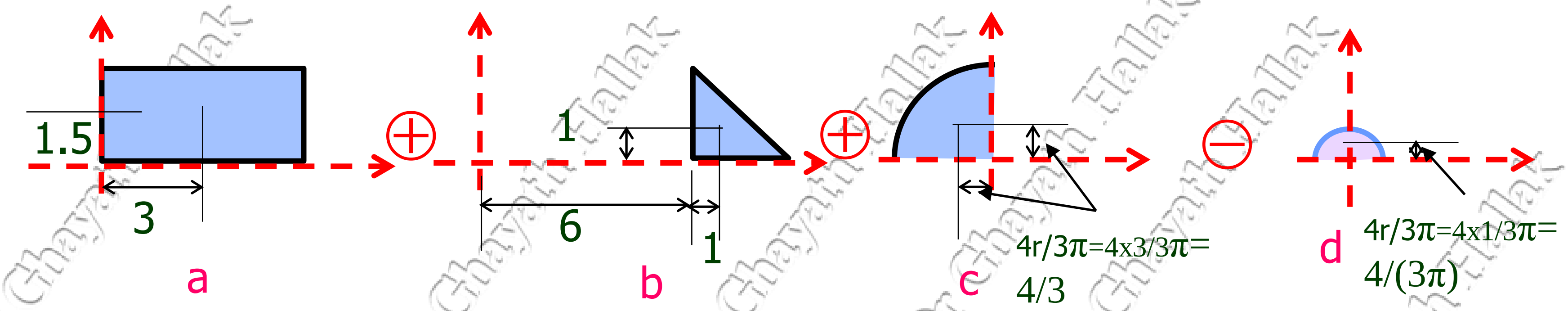
Find: The centroid of the part.



Solution:

1. This body can be divided into the following pieces: rectangle (a) + triangle (b) + quarter circular (c) – semicircular area (d)





Segment	Area A (in^2)	$\tilde{x}$ (in)	$\tilde{y}$ (in)	$A\tilde{x}$ (in^3)	$A\tilde{y}$ (in^3)
Rectangle Triangle Q. Circle Semi-Circle	18 4.5 $9\pi/4$ $-\pi/2$	3 7 $-4(3)/(3\pi)$ 0	1.5 1 $4(3)/(3\pi)$ $4(1)/(3\pi)$	54 31.5 -9 0	27 4.5 9 - 2/3
Σ	28.0			76.5	39.83

$$x_c = (\Sigma \tilde{x} A) / (\Sigma A) = 76.5 \text{ in}^3 / 28.0 \text{ in}^2 = 2.73 \text{ in}$$

$$y_c = (\Sigma \tilde{y} A) / (\Sigma A) = 39.83 \text{ in}^3 / 28.0 \text{ in}^2 = 1.42 \text{ in}$$