# Modelling of AC susceptibility of thin superconducting disk in perpendicular magnetic fields

**Maan Salim[1]**

Received 18/01/2015
Accepted 14/05/2015

## ABSTRACT

**A theoretical model for the real and imaginary parts of the susceptibility of superconducting thin disk as a function of a perpendicular applied magnetic field were derived. In order to do the calculation according to this model, matlab codes were written. The calculation was done with the assumption that the critical current density is independent of the applied magnetic field (The Bean critical state Model). The real and imaginary parts of susceptibility was then calculated at different films thickness and at different critical current densities. The calculation was repeated with the assumption that the critical current densities are magnetic field dependent by using the concepts of Kim and Exponential models. It was found that the magnetic field of full penetration increases linearly as the disk thickness increases, $H_p = 0.9267d + 1.0984$ for the Bean Model, $H_p = 17.772d + 132.18$ for the Exponential model, and $H_p = 19.363d - 159.88$ for the Kim model. Moreover, the magnetic field of full penetration increases linearly as the critical current densities increases. The results of using the ac susceptibility model showed agreement with some of published works.**

**Keywords:** Modelling magnetic susceptibility, superconducting thin disk, perpendicular magnetic field to superconducting specimen, critical current densities, susceptibility geometry dependent, demagnetizing field.

[1] Assistant Prof., Department of Physics, Faculty of Science, Damascus University, Damascus, Syria

# إيجاد نموذج نظري للطواعية المغناطيسية المتناوبة لغشاء ناقل فائق رقيق بشكل قرص في حقول مغناطيسية عمودية على سطحه

**معن سليم**[(1)]

تاريخ الإيداع 2015/01/18
قبل للنشر في 2015/05/14


## الملخص

**تم إيجاد نموذج نظري للقسمين الحقيقي والتخيلي للطواعية المغناطيسية المتناوبة للنواقل الفائقة على شكل قرص رقيق كتابع لشدة حقل مغناطيسي عمودي على سطحه. من أجل إجراء حسابات القسمين الحقيقي والتخيلي للطواعية المغناطيسية المتناوبة للنواقل الفائقة وفق هذا النموذج كُتب عدد من الكودات باستخدام لغة الماتلاب. في البداية أجريت حسابات القسمين الحقيقي والتخيلي للطواعية المغناطيسية المتناوبة للنواقل الفائقة بفرض أن كثافة التيار مستقلة عن الحقل المغناطيسي المطبق (نموذج بين). بعدها أُجريت حسابات القسمين الحقيقي والتخيلي للطواعية المغناطيسية المتناوبة للنواقل الفائقة بدلالة ثخانة القرص و بدلالة كثافة التيار. أُعيدت الحسابات المذكورة آنفاً بفرض أن كثافة التيار الحرج تابع للحقل المغناطيسي المطبق وفقاً للنموذج الآسي ووفقاً لنموذج كيم. وقد وجد أن الحقل المغناطيسي الداخل الى وسط العينة يزداد بشكل خطي مع ثخانة القرص؛ $H_p = 0.9267d + 1.0984$ من أجل نموذج بين؛ $H_p = 17.772d + 132.18$ من أجل النموذج الأسي؛ و $H_p = 19.363d - 159.88$ من أجل نموذج كيم. فضلاً عن ذلك، وجد أن الحقل المغناطيسي الداخل إلى وسط العينة يزداد بشكل خطي مع زيادة كثافة التيار الحرج. أخيراً وجد أن النتائج التي تم الحصول عليها باستخدام هذا النموذج تتوافق مع بعض النتائج المنشورة.**


**الكلمات المفتاحية:** نمذجة الطواعية المغناطيسية، غشاء رقيق على شكل قرص ناقل فائق، حقل مغناطيسي معامد لسطح الناقل الفائق، كثافة التيار الحرج، تابعية الطواعية على هندسة العينة، الحقل المغناطيسي المعاكس للمغنطة.

[(1)] مدرس، قسم الفيزياء، كلية العلوم، جامعة دمشق، سورية.

## Introduction

The ac susceptibility of superconducting material as a function of applied magnetic field parallel to its longest axis is extensively investigated. This was straight forward due to the fact that the demagnetizing field is negligible in this case where the demagnetisation factor is N=0. This has permitted analytical solutions to be obtained for slab and cylindrical geometries, for example. For the case of thin films the measurement with applied field parallel to the surface of the sample is not practical because of the following two reasons. Firstly, the measured voltage signals obtained for the parallel field measurements are very weak because of the small amount of material exposed to the magnetic fields. Secondly, the parallel alignment is very difficult to achieve because even small deviations from parallel increase the measured voltage signal and a demagnetisation correction should be taken into account.

The more practical case is with the field applied perpendicular to the specimen's plane, which gives a much stronger voltage signal and hence magnetisation. However, the analysis for this case is more complex due to the strong contribution of the demagnetizing field, the demagnetizing factor here takes a largest value N=1. As a result, the solution of the critical state equation, $curl\ H=J_c$ can only be obtained numerically [1]. This work focus on the case where the magnetic field is applied perpendicular to the thin film surface of disk geometry.

## Critical State Models

The superconducting materials with strong pinning in high magnetic field were first introduced by Bean [2], [3]. This model assumes the demagnetizing field effects are neglected; this applies to the long geometry of the specimen in a parallel applied magnetic field. The main assumption of this model is that the current density inside the superconductor is either zero or has the critical magnitude $J_c$ which is magnetic field independent. The zero current density $J_c = 0$ corresponds to the Meissner state where the magnetic flux has not penetrated into the sample and the magnetic induction inside the sample $B=0$. The sample is in the critical state when the current density reaches the critical current density. Kim *et al* [4] modified the Bean model by taking into account the field dependent critical current density as follows, $J_c(T,B) = J_0(T)/[1 + (B/P)]$. Fietz *et al* [5] have introduced the exponential model which also takes into account the field dependence of the critical current density $J_c(T,B) = J_0(T)\exp\left(-\frac{B}{P}\right)$. Where $J_0(T)$ is the critical current density at zero magnetic field and at temperature T; $B$ is the local flux density; and $P$ is a model-dependent parameter.

## Magnetic Moment

The magnetic moment of a type-II superconductor is often calculated using the Bean model. Features of this model which were addressed by Brandt [6] are as follows: When the applied field $H_a$ is cycled between limits $-H_0$ and $+H_0$, the virgin curve $m(H_a)$ corresponding to the increase of $H_a$ from zero determines the full hysteresis loop of $m(H_a)$. The branches of the hysteresis magnetic moment $m_\uparrow$ and $m_\downarrow$ in increasing and decreasing magnetic filed are given as [7],[8], [9], [10]:

$$m_\uparrow = -m(H_0) + 2m\left(\frac{H_0 + H_a}{2}\right) \qquad (1)$$

$$m_\downarrow = m(H_0) - 2m\left(\frac{H_0 - H_a}{2}\right) \qquad (2)$$

Where $H_a$ is the ac magnetic field and is given by $H_a = H_0\exp(i\emptyset)$, which it can be written in terms of complex components as $H_a(t) = H_0\cos(\emptyset)$ for the real part and $H_a(t) = H_0\sin(\emptyset)$ for the imaginary part, $H_0$ .is the field magnitude and $\emptyset$ is the phase. The magnetic moment of a circular disk of radius $a$ and thickness $d$ $(d \ll a)$ in transverse magnetic filed is given by [11]:

$$m_{disk}(h) = -J_c d a^3 \frac{2}{3}\left(\cos^{-1}\frac{1}{\cosh(h)} + \frac{\sinh|h|}{\cosh^2(h)}\right) \qquad (3)$$

With $h = \frac{H_a}{H_c}$, where $H_c = \frac{J_c d}{2}$ is the critical magnetic field for the superconducting disk, and $J_c$ is the critical current density of the disk.

## Results and Discussions

Using the above equation the ac susceptibility model was derived. From equation (2) and for the type II superconducting samples, the maximum real part of the magnetic moment corresponds to $(\emptyset = 0)$ hence $H_a = H_0 cos(\emptyset) = H_0$ , therefore the real part of equation (2) can be written as:

$$m'_{\downarrow} = m(H_0) \qquad (4)$$

On the other hand, since there is a 90 degree phase shift between the in- and out-of-phase, the phase for the imaginary part magnetic moment is $\emptyset = 90$ hence $H_a = H_0 sin(\emptyset) = 0$ , therefore using equation (2) the imaginary part of the magnetic moment can be written as:

$$m''_{\downarrow} = m(H_0) - 2m\left(\frac{H_0}{2}\right) \qquad (5)$$

Substituting equation (3) into equations (4) and (5) gives the real and imaginary parts magnetic moment of the disk respectively as follows:

$$m'_{\downarrow}(h) = -J_c d a^3 \frac{2}{3}\left[\cos^{-1}\frac{1}{\cos h(h)} + \frac{\sinh|h|}{\cosh^2(h)}\right] \qquad (6)$$

and

$$m''_{\downarrow}(h) = J_c d a^3 \frac{2}{3}\left[-\left(\cos^{-1}\frac{1}{\cos h(h)} + \frac{\sinh|h|}{\cosh^2(h)}\right) + 2\left(\cos^{-1}\frac{1}{\cos h\left(\frac{h}{2}\right)} + \frac{\sinh\left|\frac{h}{2}\right|}{\cosh^2\left(\frac{h}{2}\right)}\right)\right] \qquad (7)$$

Where $h = \frac{H_0}{H_c}$, and $H_c = \frac{J_c d}{2}$. Using the expression $\chi = \chi' + i\chi'' = \frac{m}{H_0 V}$; one can calculate the real and the imaginary parts of the susceptibility respectively as follows $\chi' = \frac{m'}{VH_0}$ and $\chi'' = \frac{m''}{VH_0}$. Where $V = \pi a^2 d$ is the disk volume. Hence the real and imaginary part of the susceptibility is given respectively by:

$$\chi' = -\frac{\chi_0}{2h}\left[\left(\cos^{-1}\frac{1}{\cos h(h)} + \frac{\sinh|h|}{\cosh^2(h)}\right)\right] \qquad (8)$$

$$\chi'' = \frac{\chi_0}{2h}\left[-\left(\cos^{-1}\frac{1}{\cos h(h)} + \frac{\sinh|h|}{\cosh^2(h)}\right) + 2\left(\cos^{-1}\frac{1}{\cos h\left(\frac{h}{2}\right)} + \frac{\sinh\left|\frac{h}{2}\right|}{\cosh^2\left(\frac{h}{2}\right)}\right)\right] \quad (9)$$

Where $\chi_0 = \frac{8a}{3\pi d}$. Equations (8) and (9) give the theoretical model for the real and imaginary parts of the susceptibility of superconducting thin disk as a function of perpendicular applied magnetic field.

In this work matlab codes were written in order to calculate the real part and imaginary part susceptibility using this model (Equations (8) and (9)). The first code was used to calculate the real and imaginary parts of the susceptibility as a function of the applied magnetic field at different films thickness. The parameters used to run the first code were: The radius of the disk was $a = 0.5 \times 10^{-2}$ m. The critical current of the disk was assumed independent of the applied magnetic field (Bean Model) and set to the value of $J_c = 10^9 \frac{A}{m^2}$, and the thickness of the films were changed from 100 nm to 2000 nm. Figure (1) shows the real and imaginary part of the susceptibility as a function of applied magnetic field and at different film thickness. The values of d from left to the right are 100nm- 200nm- 300nm- 400nm- 500nm- 600nm- 700nm- 800nm- 900nm- 1000nm. As well known the real part of the susceptibility is a measure of the screening ability and the imaginary part (the peaks) is a measure of the dissipation due to the flux motion. From figure (1), the peak in the imaginary part of the susceptibility is shifted to higher magnetic field as the thickness of the disk is increased. At the peak of the imaginary part of the susceptibility the magnetic field penetrates into the sample centre and hence the dissipation has maximum value. The magnetic filed corresponding to the peak of the imaginary part of the susceptibility is called the magnetic field of full penetration. For clarity figure (2) plots the imaginary part of the susceptibility which is taken from Figure (1). However, the real part of the susceptibility at lower value of d increases sharply as the applied magnetic field increases and at higher value of d increases gradually with increasing the magnetic fields. There is no experimental data or theoretical data match results obtained in this work for specific geometry of thin superconducting disk in perpendicular magnetic field or match the conditions used in this work. However, results of this work in general match the behaviour of the magnetic susceptibility of thin superconductors in perpendicular magnetic fields with different conditions and different calculation approaches. For examples, D. V. Shantsev el al [12] and Chen D.-X. et al [13], [14] have different approach to calculate the ac susceptibility and our results in general are in agreements with their works. On the other hand, results are also in general consistent and in agreement with the experimental data by Pérez I. et al [15].

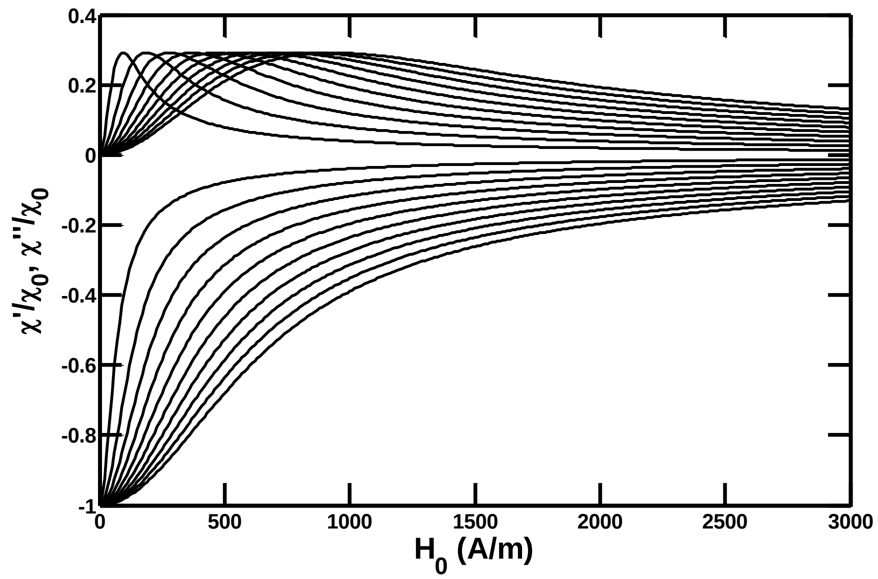


**Figure (1): The real and the imaginary part of the susceptibility as a function of applied magnetic field and at different film thickness (Bean Model). The values of d from left to the right are 100nm- 200nm- 300nm-400nm-500nm-600nm-700nm-800nm-900nm-1000nm,** $J_c = 1 \times \frac{10^9 A}{m^2}$

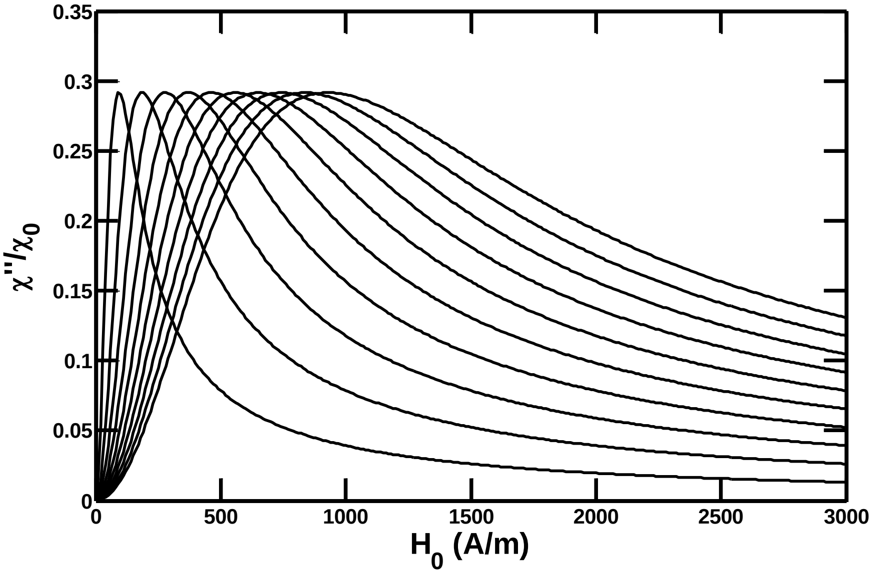


**Figure (2): The imaginary part of the susceptibility as a function of applied magnetic field and at different film thickness (Bean Model). The values of d from left to the right are 100nm- 200nm - 300nm - 400nm - 500nm - 600nm - 700nm - 800nm- 900nm-1000nm, $J_c = 1 \times \frac{10^9 A}{m^2}$ .**

Figure (3) shows the magnetic field of full penetration as a function of the disk thickness. The critical current was firstly assumed to be magnetic field independent (The Bean model). Then the matlab code was amended to take into account the dependent of critical current density on the magnetic field according to the Exponential model $J_c(T,B) = J_0(T)\exp\left(-\frac{B}{P}\right)$ and Kim model $J_c(T,B) = J_0(T)/[1 + (B/P)]$. Where for Exponential model the constant P was taken to be 1 (P=1), and for Kim model (P=1). For both Exponential and Kim models the critical current density $J_c(T,B)$ and the critical current density at zero field $J_0(T)$ of the thin disk were taken as follows: $J_c(T,B) = 1 \times \frac{10^9 A}{m^2}$ and

$J_0(T) = 2 \times \frac{10^{10}A}{m^2}$. It is clear that the values of magnetic field of full penetration are linearly proportional to the disk thickness. Fitting these curves gives the following expressions: $H_p = 0.9267d + 1.0984$ for the Bean Model, $H_p = 17.772d + 132.18$ for the Exponential model, and $H_p = 19.363d - 159.88$ for the Kim Model. (Note: the results shown in figure (1) for disk thickness from 100 nm to 1000 nm. Because of the similarity of the results in figure (1), the rest of results of disk thickness from 1000 nm to 2000 nm and from 10 nm to 100 nm are not presented. The advantages of the later results are to determine the magnetic field of full penetration at those values of film thickness see figure (3)).

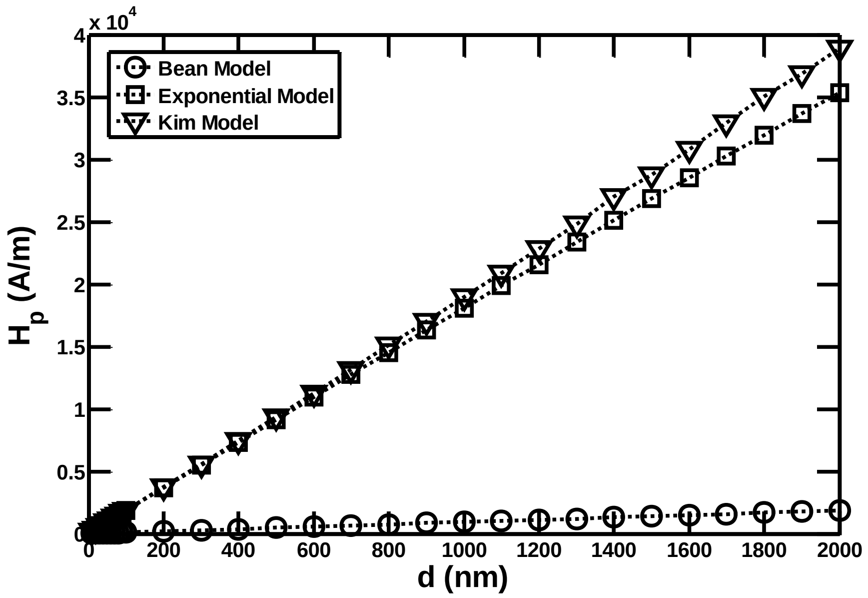


**Figure (3): The magnetic field of full penetration as a function of film thickness according to Bean Model, Exponential Model (p=1), and Kim Model (p=1), $J_c(T,B) = 1 \times \frac{10^9A}{m^2}$ and $J_0(T) = 2 \times \frac{10^{10}A}{m^2}$.**

A second matlab code was written in order to calculate the real and the imaginary part of the susceptibility using equations (8) and (9) as a function of applied magnetic field and at different critical current densities $J_c$ see figure (4). The values of $J_c$ from left to the right are $\frac{[(0.25-0.5-0.75-1-1.25-1.5-1.75-2-2.25-2.5)\times 10^9]A}{m^2}$, the film thickness was 200nm, the code were run for different thickness (see figure (6)). The critical current densities were assumed independent of applied magnetic field (Bean Model). Taking into account Kim or exponential model in this case is un-useful because that both of magnetic field of full penetration and the critical current densities are magnetic field dependent, hence there is no peak shift in the imaginary part of the susceptibility due to changing values of critical current densities.

From Figure (4), it can be noticed that the peak in the imaginary part of the susceptibility shift to higher magnetic field as the critical current density was increased. The code was run again to include the real and the imaginary part of the susceptibility at higher values of critical current densities up to $25\times 10^9 \frac{A}{m^2}$. However, the real part of the susceptibility at lower value of critical current density increases sharply as the applied magnetic field increases and at higher value of critical current density increases gradually with increasing the magnetic fields. Figures (5) shows a plot of magnetic field of full penetration as a function of the critical current densities at different film thickness. It is clear from this figure that the magnetic field of full penetration is linearly proportional to the critical current densities, fitting this curve gives the expression for d=100 nm $H_p = 9.2689\times \mathbf{10^8} J_c - \mathbf{0.38274}$, $H_p = 1.8317\times \mathbf{10^7} J_c + \mathbf{4.527}$ for d=200 nm, $H_p = 4.6294\times \mathbf{10^7} J_c + \mathbf{0.50671}$ for d=500 nm, and $H_p = 9.2724\times \mathbf{10^7} J_c - \mathbf{5.9014}$ for d=1000 nm. From figure (3) and figure (5) one can conclude that the magnetic field of full penetration is directly proportional to the disk thickness and to the critical current densities.

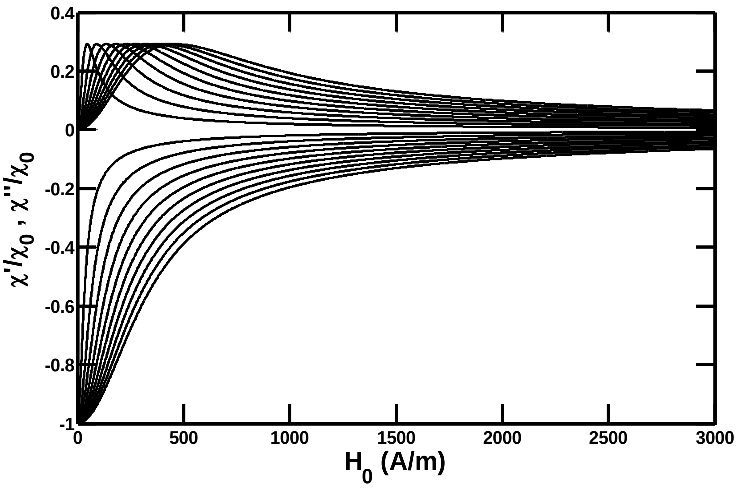


**Figure (4): The real and the imaginary part of the susceptibility as a function of applied magnetic field and at different critical current densities. The values of $J_c$ from left to the right are $\frac{[(0.25-0.5-0.75-1-1.25-1.5-1.75-2-2.25-2.5)\times 10^9]A}{m^2}$, the film thickness is 200nm**

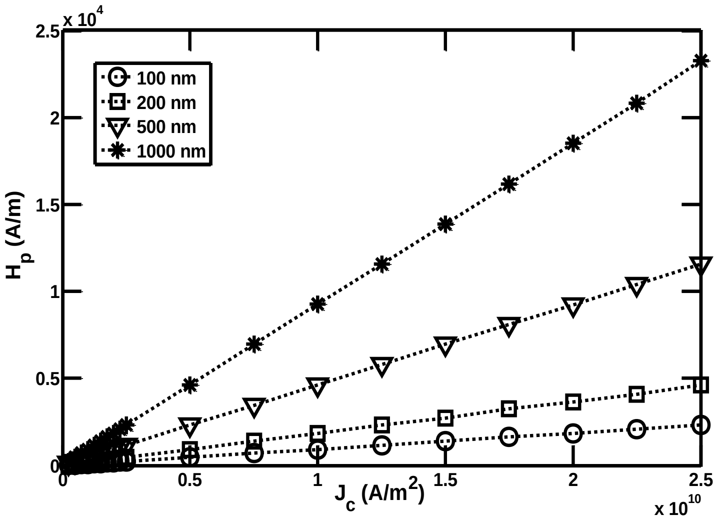


**Figure (5): The magnetic field of full penetration as a function of critical current density according to Bean Model, the film thickness are 100nm, 200nm, 500 nm, 1000 nm.**

## Conclusion

A theoretical model for the real and imaginary parts of the susceptibility of thin disk in perpendicular applied magnetic field gives us very interesting information regarding the dependency of the magnetic field of full penetration $H_p$ on the film thickness and on the critical current density $J_c$ of the specimen. $H_p$ was found to linearly increase with increasing both of film thickness $d$ and the critical current densities $J_c$. Therefore one can write an expression for $H_p$ in the form of: $H_p = aJ_c d + b$ where $a$ and $b$ are constant.

# REFERENCES

**[1] Xing, W. Heinrich, B. Chrzanowski, J. Irwin, J. C. Zhou, H. Cragg, A. and Fife, A. A., 1993. Determination of critical current densities of $YBa_2Cu_3O_{7-\delta}$ thin films from AC susceptibility measurements, Physica C 205, p 311.**

**[2] Bean, C. P. and Livingston, J. D., 1964. Magnetic Properties of Superconducting Lead-Base Alloys, Phys. Rev. Lett. 12, p 14.**

**[3] Bean, C.P., 1962. Magnetisation of high field superconductors, Phys. Rev. Lett. 8 250; 1964, Rev. Mod. Phys. 36, p 31.**

**[4] Kim, Y. B. Hempstead, C. F. and Strnad, A. R., 1962. Critical persistent currents in hard super conductors, Phys. Rev. Lett. 9, p 306: Kim, Y. B. Hempstead, C. F. and Strnad, A. R., 1963. Phys. Rev. 129, p 528.**

**[5] Fietz, W. A. Beasley, M. R. Silcox, J. and Webb, W. W., 1964. Magnetization of Superconducting Nb-25%Zr Wire, Phys. Rev. 136, A335.**

**[6] Brandt, E. H., 1997. Susceptibility of superconductor disks and rings with and without flux creep, Phys. Rev. B 55, p 14513.**

**[7] Gugan, D., 1994. A heuristic, general, field-dependent critical-current model for the magnetic properties of type-II superconductors, Physica C233, p 165.**

**[8] Brandt, E. H., 1995. Electric field in superconductors with rectangular cross section, Phys. Rev. B 52, p 15442.**

**[9] Brandt, E. H., 1998. Magnetic response of superconductors in various geometries, Supercond, Supercond. Sci. Technol. 11, p 921.**

**[10] Brandt, E. H. Indenbom, M. V., 1993. Type-II-superconductor strip with current in a perpendicular magnetic field, Phys. Rev. B 48, p12893.**

**[11] Zadro, K. Babic, E. Liu, H. K. Dou, S. X., 1996. Irreversible magnetization and critical currents in silver-sheathed (Bi, Pb)$_2Sr_2Ca_2Cu_3O_{10}$ + y tape, Phys. Stat. Sol. (a) 157, p 427.**

**[12] Shantsev, D. V. et al, 2000. Thin superconducting disk with field-dependent critical current: Magnetization and ac susceptibilities, Phys. Rev. B, 61(14): 9699.**

**[13] Chen, D.-X. Navau, C. Del-Valle, N. Sanchez, A., 2010. Analytic expressions for critical-state ac susceptibility of rectangular superconducting films in perpendicular magnetic field, Physica C 470, p 89–94.**

**[14] Chen, D-X. et al, 2011. Perpendicular ac susceptibility and critical current density of distant superconducting twin films, Supercond. Sci. Technol. 24, p 075004.**

**[15] Pérez, I. Sosa, V. Gamboa, F., 2010. Study of higher-order harmonics of complex ac susceptibility in $YBa_2Cu_3O_{7-\delta}$ thin films by the mutual inductive method, Physica C 470 p. 2061–2066.**