The Eigenvalues for Preconditioned System in Interior Points Method for Quadratic Programming

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ABSTRACT

Krylov subspace methods are one of the strongest iterative approach, which are used to solve large scale linear systems. These methods are based on a general type of projection process. As at each iteration of interior points method (IPM) at least one linear system has to be solved, where the main computational effort of IPMs consist in the computational of these linear systems. Therefore, we suggest to use Krylov subspace methods to solve these linear systems.

In the paper (Al-Jeiroudi, 2011), it has been introduced a new preconditioner for the augmented system, which arises from IPM for quadratic programming. Since, in most cases Krylov subspace methods require preconditioner to improve the global convergence. The eigenvalues play critical role in the convergence analysis of Krylov subspace methods. Therefore, in this paper we are going to focus our attention to study the eigenvalues of the preconditioned augmented system, which are preconditioned by the previous preconditioner.

In this paper, we introduced new theorem, which studies the eigenvalues behaviour of the preconditioned system, and we show that, the eigenvalues are bounded away from zero. In addition, they are equal to one exactly or one plus a positive number. Moreover, they are well clustered and consequently this play an important role in the convergence speed.

Keywords: Interior Point Methods, Quadratic Programming, Krylov subspace methods, Preconditioner, Eigensystem, Convergence of iterative methods.

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القيم الذاتيه للنظام المكيف في طريقة النقاط الداخلية للبرامج التربيعية

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الملخص

تعدُّ طرائق فضاءات كليروف الجزئية من أهم الطرائق التكرارية المناسبة لحل الأنظمة الخطية الضخمة وأقواها. تعتمد هذه الطرائق بشكل عام على فكرة الإسقاطات. حاول هذا البحث استخدام طرائق فضاءات كليروف لحل الأنظمة الخطية الناتجة عن طريقة النقاط الداخلية، ومبرر ذلك أنه في كل تكرار من تكرارات طريقة النقاط الداخلية هناك نظام خطي واحد على الأقل بحاجة لحل؛ ممَّا يعني أن حل مجموع الأنظمة سيأخذ معظم الوقت اللازم لتنفيذ البرنامج ككل.

في المقال (Al-Jeiroudi, 2011) قُدِّم مكيف جديد للنظام الخطي في طرائق النقاط الداخلية للبرامج التربيعية، ووُجِّه البحث في هذه المقالة لدراسة القيم الذاتية لهذا النظام المكيف لأن معظم طرائق كليروف تتطلب مكيفاً لتحسين هذه الطرائق للحل وتسريع تقاربها، كما تؤدي القيم الذاتية لمصفوفات الأنظمة دوراً مهماً وأساسياً في تقارب هذه الطرائق.

قدمنا في هذا البحث مبرهنة جديدة تدرس سلوك القيم الذاتيه للأنظمة المكيفة وتبيّن أن هذه القيم لا تقترب من الصفر، وبأنَّها إمَّا أن تساوي الواحد تماماً أو مساوية للواحد مضافاً إليها قيمة موجبة تماماً فضلاً عن أن هذه القيم مجمعة مما يسرع التقارب.

الكلمات المفتاحية: طرائق النقاط الداخلية، البرامج التربيعية، فضاءات كليروف الجزئية، المكيفات، الفضاءات الذاتية، تقارب الطرائق التكرارية.

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1 Introduction

The quadratic programming problem, in general, has the form (Wright, 1997):

$$\min \quad c^T x + \frac{1}{2} x^T Q x$$
$$s.t. \quad Ax = b,$$
$$x \geq 0.$$

Where $A \in R^{m \times n}$, $Q \in R^{n \times n}$. We assume that $m \leq n$ and A has full row rank and Q is a symmetric positive definite matrix.

We will focus, in this paper, on the augmented system, which arises from interior points method (IPM) for quadratic programming problems. This system is as follow, (Andersen *et al.*, 1996 and Wright, 1997):

$$\begin{bmatrix} -(D^{-2} + Q) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \qquad (1)$$

Where $D = X^{1/2} S^{-1/2}$ and X and S are diagonal matrices in $R^{n \times n}$ with the elements of vectors $x \in R^n$ and $s \in R^n$ respectively on the diagonal, µ is the average complementarity gap $m = x^T s / n$, (Wright, 1997).

f and g are given as follows:

$$f = c - A^T y + Qx - mX^{-1} e \quad \text{and} \quad g = b - Ax,$$

where $c \in R^n$, $y, b \in R^m$.

The solution of (1) dominates on the computational effort of the entire IPM algorithm. Especially, it must be solved at each iteration of IPM algorithm during the execution.

In (Al-Jeiroudi, 2011) it has been introduced the following vectors t_s and t_y which define:

$$t_s = X^{1/2} S^{-1/2} \Delta s, \qquad t_y = \Delta y$$

That leads to the following indefinite augmented system of linear equations

$$\begin{bmatrix} (I + DQD) & DA^T \\ AD & 0 \end{bmatrix} \begin{bmatrix} t_s \\ t_y \end{bmatrix} = \begin{bmatrix} \bar{f} \\ \bar{g} \end{bmatrix} \qquad (2)$$

Where $\bar{f} = X^{1/2} S^{-1/2} (c - A^T y - s + mQS^{-1}e)$ and $\bar{g} = -b + mAS^{-1}e$.

In this paper, we will suggest using one of Krylov subspace methods to solve the system (2) preconditioned by the preconditioner, which it has been proposed in (Al-Jeiroudi, 2011). Usually, the systems (1) and hence (2) are very huge. As a result, preconditioned the system is essential to improve the convergence property of the system.

The objective here is to study the eigenvalues behaviour of our specific preconditioned system, preconditioned by the preconditioner, which is proposed in (Al-Jeiroudi, 2011). We will also show that the eigenvalues are maintained away from zero and they are well clustered.

Many studied tackle the issue of convergence analysis and eigenspace of Krylov subspace methods (Saad, 2003; Van der Vorst, 2003; Greenbaum, 1997 and Kelley, 1995). The convergence of Krylov subspace methods is determined by the distribution of the eigenvalues of the coefficient matrix. It is often desirable for the number of distinct eigenvalues to be small so that the global convergence is guaranteed to occur quickly (Trefethen *et al.,* 2005; Greenbaum, 1997 and Nachtigal *et al.,* 1992). In addition, clustered spectrum away from zero often results in rapid convergence (Benzi *et al.,* 2005).

3 Block triangular preconditioner

In this section, we will remind the reader by the preconditioner of system (2), which it has been mentioned in the paper (Al-Jeiroudi, 2011). In order to study the behaviour of the eigenvalues of the preconditioned matrix.

Many researchers provide a certain class of preconditioners, ones try to guess a non-singular sub-matrix of A, which is called the basis of A (Al-Jeiroudi *et al.,* 2009 and 2008; Bergamaschi *et al.,* 2007 Chai *et al.,* 2007; Dollar *et al.,* 2006, 2005 and 2004; Keller *et al.,* 2000 and Gill *et al.,* 1992).

We are going to mention here the preconditioner for the augmented system (2), that we want to study its eigenvalues behaviour. This preconditioner has the form:

$$p = \begin{bmatrix} I + D_B Q_{11} D_B & D_B Q_{12} D_N & D_B B^T \\ D_N Q_{21} D_B & I & 0 \\ B D_B & 0 & 0 \end{bmatrix} \qquad (3)$$

Where the following matrix:

$$K = \begin{bmatrix} (I + DQD) & DA^T \\ AD & 0 \end{bmatrix},$$

in the augmented system is written:

$$K = \begin{bmatrix} I + D_B Q_{11} D_B & D_B Q_{12} D_N & D_B B^T \\ D_N Q_{21} D_B & I + D_N Q_{22} D_N & D_N N^T \\ B D_B & N D_N & 0 \end{bmatrix}.$$

Hence, the matrix A is divided into $[B \quad N]$, where B is a $m \times m$ full rank matrix the columns of A which is corresponding to the small elements of D. The matrix N is the rest columns of A.
Similarly, the matrix Q is also divided as follows:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

where Q_{11}, Q_{12}, Q_{21} and Q_{22} are matrices of the size $m \times m$, $n - m \times m$, $m \times n - m$ and $n - m \times n - m$ respectively.

4 The Eigenvalues of the Preconditioned system:

In the following theorem, we show that the eigenvalues of the preconditioned matrix are: one with multiplicity $2m - r$, where r is the rank of N, or one plus the eigenvalues of the symmetric positive definite matrix, which is positive real number, and one plus complex number with positive real part. Therefore, the spectrum of the preconditioned matrix is well clustered.

The preconditioned matrix $P^{-1}K$, where the inverse of the preconditioner P is given as follows (Al-Jeiroudi, 2011):

$$p^{-1} = \begin{bmatrix} 0 & 0 & {D_B}^{-1}B^{-1} \\ 0 & I & -D_N Q_{21} B^{-1} \\ B^{-T}{D_B}^{-1} & -B^{-T}Q_{12}D_N & B^{-T}(-D_B^{-2} - Q_{11} + Q_{12}D_N^2 Q_{21})B^{-1} \end{bmatrix}.$$

Therefore the preconditioned matrix $P^{-1}K$ becomes:

$$p^{-1}K = \begin{bmatrix} I & {D_B}^{-1}B^{-1}ND_N & 0 \\ 0 & I + D_N(Q_{22} - B^{-1}N)D_N & D_N N^T \\ 0 & -B^T Q_{12}D_N^2 Q_{22}D_N + B^{-T}(-D_B^{-2} - Q_{11} + Q_{12}D_N^2 Q_{21})B^{-1}ND_N & I + B^T Q_{12} D_N^2 N^T \end{bmatrix}.$$

Theorem: The eigenvalues of the preconditioned matrix $P^{-1}K$ are exactly one (with multiplicity $2m - r$, or maintained away from zero which are either real number of the form

$$l = 1 + \frac{v_N^T (D_N Q_{22} D_N + eI)v_N}{v_N^T v_N} \quad \text{where } e \text{ is a small positive constant}$$

or complex number of the form

$$l = 1 + \frac{f^T D_N Q_{22} D_N f + h^T D_N Q_{22} D_N h + 2if^T (C - C^T)h}{f^T f + h^T h}$$

Where $C = D_N N^T B^{-T} Q_{12} D_N$ and $v_N = f + ih.$

Proof: Let v be the eigenvector of $P^{-1}K$ corresponding to the eigenvalue l, that is $P^{-1}Kv = lv.$ Let $l = 1 + t$ and performing the usual partitioning $v = [v_B, v_N, v_y].$

The eigensystem can be rewritten as $Kv = (1 + t)Pv.$ By substitution, we get the following:

$$\begin{bmatrix} I+D_BQ_{11}D_B & D_BQ_{12}D_N & D_BB^T \\ D_NQ_{21}D_B & I+D_BQ_{22}D_N & D_NN^T \\ BD_B & ND_N & 0 \end{bmatrix}\begin{bmatrix} v_B \\ v_N \\ v_y \end{bmatrix}$$

$$=(1+t)\begin{bmatrix} I+D_BQ_{11}D_B & D_BQ_{12}D_N & D_BB^T \\ D_NQ_{21}D_B & I & 0 \\ BD_B & 0 & 0 \end{bmatrix}\begin{bmatrix} v_B \\ v_N \\ v_y \end{bmatrix}$$

which yields:

$$t[(I+D_BQ_{11}D_B)v_B + D_BQ_{12}D_Nv_N + D_BB^Tv_y]=0, \qquad (1)$$

$$D_NQ_{22}D_Nv_N + D_NN^Tv_y = tD_NQ_{21}D_Bv_B + tv_N, \qquad (2)$$

$$ND_Nv_N = tBD_Bv_B. \qquad (3)$$

Equation (1) is true for $t=0$ or

$$(I+D_BQ_{11}D_B)v_B + D_BQ_{12}D_Nv_N + D_BB^Tv_y = 0.$$

The first case $t=0$.

By the substitution $t=0$ in (2) and (3), we will have:

$$D_NQ_{22}D_Nv_N + D_NN^Tv_y = 0, \qquad (4)$$

$$ND_Nv_N = 0. \qquad (5)$$

Now, we will discuss a number of cases depending on v_B, v_N and v_y.

a. The case of $v_N=0$ and $v_y=0$, the two equations (4) and (5) are true. That gives the eigenvector $v=[v_B,0,0]$ associated with the unit eigenvalue with multiplicity m, Because there is a way to find easily m linearly independent vectors v_B.

b. The case of $v_B=0$ and $v_N=0$, the eigenvector $v=[0,0,v_y]$ associated with the unit eigenvalue with multiplicity $m-p$. Because it can be found $m-p$ linearly independent vectors.

The second case:

$$(I+D_BQ_{11}D_B)v_B + D_BQ_{12}D_Nv_N + D_BB^Tv_y = 0. \qquad (6)$$

From (3) we get $tv_B = D_B^{-1}B^{-1}ND_N v_N$. By substituting this in equation (2), it can be found:

$$D_N Q_{22} D_N v_N + D_N N^T v_y - D_N Q_{21} B^{-1} N D_N v_N = t v_N . \qquad (7)$$

Now, we will discuss a number of cases depending on v_B, v_N and v_y.

a. Let us choose v_N such that:

$$(eI + D_N Q_{21} B^{-1} N D_N) v_N = D_N N^T v_y , \qquad (8)$$

Where e is either $e = 0$ when $D_N Q_{21} B^{-1} N D_N$ is a nonsingular or $e \in (0,d)$ when $D_N Q_{21} B^{-1} N D_N$ is a singular, where $d = \min_i \{|x_i| : x_i \neq 0\}$, x_i are all eigenvalues of $D_N Q_{21} B^{-1} N D_N$.

The choice of e guarantees that the matrix $(eI + D_N Q_{21} B^{-1} N D_N)$ is non-singular. For $v_y \neq 0$ and $D_N N^T v_y \neq 0$, conclude that $v_N \neq 0$.

By substituting (8) in (7), we will find:

$$(eI + D_N Q_{22} D_N) v_N = t v_N .$$

t is a real positive number, since $(eI + D_N Q_{22} D_N)$ is symmetric and also positive definite.

For $v_N \neq 0$, it can be found that:

$$t = \frac{v_N^T D_N Q_{22} D_N v_N + e v_N^T v_N}{v_N^T v_N} .$$

The vector v_B will be chosen such that the equation (6) should be satisfied. That gives the eigenvector $v = [v_B, v_N, v_y]$, associated with the eigenvalue $l = 1 + t$ with multiplicity r. Because we can find r linearly independent vectors v_y such that $D_N N^T v_y \neq 0$, which is also linearly independent with the vectors v_y in case 1.b.

b. Let us choose:

$$v_y = B^{-T} Q_{12} D_N v_N .$$

By substituting this in equation (7), we get

$$D_N Q_{22} D_N v_N + D_N N^T B^{-T} Q_{12} D_N v_N - D_N Q_{21} B^{-1} N D_N v_N = t v_N .$$

Let $C = D_N N^T B^{-T} Q_{12} D_N$, so the previous equation becomes

$$D_N Q_{22} D_N v_N + C v_N - C^T v_N = t v_N. \qquad (9)$$

Because Q is a symmetric matrix.

Let us multiply the equation (9) by v_N^H. We can obtain the following equation:

$$v_N^H D_N Q_{22} D_N v_N + v_N^H C v_N - v_N^H C^T v_N = t v_N^H v_N. \qquad (10)$$

Let $v_N = f + ih$, where f and h are the real and the imaginary part of v_N respectively.

Then, the equation (10) can be rewritten as follows:

$$f^T D_N Q_{22} D_N f + h^T D_N Q_{22} D_N h + 2if^T (C - C^T) h = t\,(f^T f + h^T h).$$

For nonzero v_N, then we will have $f^T f + h^T h \neq 0.$ That leads to:

$$t = \frac{f^T D_N Q_{22} D_N f + h^T D_N Q_{22} D_N h + 2if^T (C - C^T) h}{f^T f + h^T h}.$$

and

$$l = 1 + \frac{f^T D_N Q_{22} D_N f + h^T D_N Q_{22} D_N h + 2if^T (C - C^T) h}{f^T f + h^T h}. \qquad (11)$$

Where the real part of l is given as:

$$\mathrm{Re}(l) = 1 + \frac{f^T D_N Q_{22} D_N f + h^T D_N Q_{22} D_N h}{f^T f + h^T h}.$$

$D_N Q_{22} D_N$ is symmetric positive definite matrix and $v_N \neq 0$. Therefore, $\mathrm{Re}(l) \geq 0$. That leads to l is kept away from zero. The vector v_B can be chosen such that the equation (6) is satisfied. That gives the eigenvectors $v = [v_B, v_N, v_y]$, associated with the eigenvalues (11) with multiplicity $n - m$. Because it can be found $n - m$ linearly independent vectors v_N.

We deduce from the previous studied cases that the preconditioned matrix $P^{-1}K$ has $2m - r$ unit eigenvalues (eigenvalues equal one) and the remaining eigenvalues are kept away from zero, which are either real of the form $1+\dfrac{v_N^T(D_N Q_{22} D_N v_N + eI)v_N}{v_N^T v_N}$, where the number $\dfrac{v_N^T(D_N Q_{22} D_N v_N + eI)v_N}{v_N^T v_N}$ is a positive number, since the matrix D_N is a diagonal matrix with positive elements and Q_{22} is a symmetric positive definite matrix, since Q is a symmetric positive definite too.

or complex of the form

$$1+\frac{f^T D_N Q_{22} D_N f + h^T D_N Q_{22} D_N h + 2if^T(C-C^T)h}{f^T f + h^T h},$$

where $v_N = f + ih$. The real part is as follow:

$$1+\frac{f^T D_N Q_{22} D_N f + h^T D_N Q_{22} D_N h}{f^T f + h^T h}.$$

Also, this number is equal to one plus a positive number for the same previous reasons.

That makes the eigenvalues of the preconditioned system are well clustered. In addition, the conditioned number of the preconditioned matrix is not ill conditioning and very well stable. As, it equals to the smallest eigenvalue divided by the largest eigenvalue. These features make the convergence analysis of iterative methods very good.

5 Conclusions

In this paper we showed that all eigenvalues of the preconditioned system are kept away from zero. They are exactly equal to one plus positive numbers. In addition, the eigenvalues are one of the following groups:

- Equal to one, or
- Equal to $1+\dfrac{v_N^T(D_N Q_{22} D_N v_N + eI)v_N}{v_N^T v_N}$, or
- Equal to $1+\dfrac{f^T D_N Q_{22} D_N f + h^T D_N Q_{22} D_N h + 2if^T(C-C^T)h}{f^T f + h^T h}$.

These lead to very important features of the preconditioned matrix: the eigenvalues are well clustered and the condition number of the preconditioned matrix is very well stable. The immediate result of those features is the fast convergence of the iterative methods, which we keen to get this result.

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