A Finite *BCI*-algebra of *KL*-Product

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ABSTRACT

A non-empty set X with a binary operation * and a distinguished element 0 is called a *BCI*-algebra if the following axioms are satisfied:

I) $((x*y)*(x*z))*(z*y)=0$

II) $(x*(x*y))*y=0$

III) $x*x=0$

IV) $x*y=0,\ y*x=0 \Rightarrow x=y$

for every $x, y, z \in X$.

Let be X a finite *BCI*- algebra, it is known that X is of *KL*- product if and only if the following condition is satisfied:

$$(a*e)*(0*e)=a\ ;\ \forall a\in X, \forall e\in L(X)$$

We present a necessary and sufficient condition for *BCI*-algebra X to be of *KL*- product, this condition is pure numerical, that is the number of elements of the row which is opposite to the zero element in the Cayley table of the operation * divides the number of elements in each row of the mentioned table.

Key words: *BCI*- algebra, *KL* -product.

جبر*BCI* المنتهي من النوع *KL*-جداء

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الملخص

إن مجموعة غير خالية X مزودة بعملية ثنائية * وبعنصر مميز 0 تدعى جبر -*BCI* إذا تحققت الخواص الآتية:

I) $((x*y)*(x*z))*(z*y)=0$

II) $(x*(x*y))*y=0$

III) $x*x=0$

IV) $x*y=0,\ y*x=0 \Rightarrow x=y$

وذلك مهما يكن x, y, z **من** X.

ليكنX جبر -*BCI* منتهياً، من المعروف أنX هو *KL*- جداء إذا تحقق فيه الشرط الآتي فقط:

$$(a*e)*(0*e)=a\ ;\ \forall a\in X, \forall e\in L(X)$$

نقدم في هذا البحث شرطاً لازماً وكافياً لكي يكون X جبر -*BCI* من النوع -*KL* جداء، هذا الــشرط عددي صرف وهو أن يكون عدد عناصر السطر المقابل للعنصر الصفري في جدول كايلي للعمليــة * قاسماً لعدد عناصر كل سطر في الجدول المذكور.

الكلمات المفتاحية: جبر -*BCI* ، -*KL* جداء.

Introduction

The notion of *BCK*- algebras was proposed by Y. Iami and K. Iseki in 1966.

In the same year K. Iseki [3] introduced the notion of *BCI* -algebra, which is a generalization of *BCK*- algebra.

After that, many mathematical papers have been published investigating some algebraic properties of *BCK/BCI*-algebras and their relationship with other universal structures including lattices and Boolean algebras.

1) - Basic definitions and results

Definition (1): A non-empty set X with a binary operation * and a distinguished element 0 is called a *BCI* - algebra if the following axioms are satisfied for every $x, y, z \in X$:

I) $((x*y)*(x*z))*(z*y)=0$

II) $(x*(x*y))*y=0$

III) $x*x=0$

IV) $x*y=0,\ y*x=0 \Rightarrow x=y$

Definition (2): A *BCI* -algebra X is called *BCK*-algebra if it satisfied:

V) $0*x=0\ ;\forall x \in X$

A partial ordering relation $\leq$ can be defined on *BCI* -algebra X for some $x, y \in X$ by: $x \leq y$ if and only if $x*y=0$

Remark: the axioms in definition (1) can be rewriting by using the symbol $\leq$ in the following simpler way:

I´) $(x*y)*(x*z) \leq z*y$

II´) $x*(x*y) \leq y$

III´) $x \leq x$

IV´) $x \leq y,\ y \leq x \Rightarrow x=y$

And the axiom V in definition (2):

V´) $0 \le x \;;\forall x \in X$

Definition (3):[5] An element a in BCI -algebra X is called an atom if and only if : $x * a = 0 \Rightarrow x = a \; (\forall x \in X)$.

Definition (4): [2] A BCI - algebra X is called p-semisimple when the following condition is satisfied: $0*(0*x)=x \; (\forall x \in X)$

J. Meng and X. L. Xin [6] introduced the notion of *KL*-product *BCI*-algebras.

Definition (5): Let X be BCI -algebra. If there exist *BCK*- algebra Y and p-semisimple BCI -algebra Z such that $X \approx Y \times Z$, then X is called BCI -algebra of *KL*-product.

Lemma (1): [7] An element a in BCI - algebra X is an atom if and only if $x*(x*a)=a \; (\forall x \in X)$

Notation: We shall denote the subset of all atoms in BCI -algebra X by $L(X)$

Lemma (2): [7] Let X be BCI -algebra then $L(X)=0*X$.

Theorem (1): [7] A BCI -algebra X is of *KL*- product if and only if it satisfies the condition:

$$(a*e)*(0*e)=a \;;\; \forall a \in X, \forall e \in L(X)$$

Proposition (1): [4], [1] In any BCI -algebra X the following Proprieties hold for every $x, y, z \in X$:

(1) $x*0 = x$

(2) $x*(x*(x*y)) = x*y$

(3) $(x*y)*z = (x*z)*y$

(4) $(x*y)*(z*y) \le x*z$

2) - Main Results

Definition (6): Let X be BCI -algebra, then for any element $a \in X$ the subset T_a is defined by:

$$T_a = \{x \in X : a*(a*x)=x\}$$

Lemma (3): Let X be BCI -algebra, and $a \in X$, then:

(1) $L(X) = T_0$

(2) $0, a \in T_a$

Proof:

(1)(i) $x \in T_0 \Rightarrow 0*(0*x) = x \Rightarrow x \in 0*X$, so $T_0 \subseteq 0*X$

(ii) $x \in 0*X \Rightarrow \exists y \in X ; x = 0*y \Rightarrow$

$0*(0*x) = 0*(0*(0*y)) = 0*y = x \Rightarrow x \in T_0$, So we have the other inclusion, $0*X \subseteq T_0$ and by lemma (2) $L(X) = T_0$.

(2)(i) $a*(a*0) = a*a = 0 \Rightarrow 0 \in T_a$

(ii) $a*(a*a) = a*0 = a \Rightarrow a \in T_a$

Proposition (2): Let X be BCI -algebra, then for any element $a \in X$ we have the following Proprieties:

(1) $T_a = a*X = \{a*x : x \in X\}$

(2) $T_a * a = L(X) = T_0$ $(\forall a \in X)$

(3) $T_{a*x} \subseteq T_a$ $(\forall x \in X)$

(4) $T_0 \subseteq T_a$

(5) $T_a * X = T_a$

(6) $x \in T_a \Rightarrow T_x \subseteq T_a$

(7) $T_0 = T_a$ if a is an atom.

Proof:

(1) (i) $y \in T_a \Rightarrow y = a*(a*y) \Rightarrow y \in a*X$, so we have: $T_a \subseteq a*X$

(ii) on the other hand: $y \in a*X \Rightarrow \exists x \in X ; y = a*x \Rightarrow$

$$a*(a*y)=a*(a*(a*x))=a*x=y \Rightarrow y\in T_a$$

This implies $a*X \subseteq T_a$. So, $T_a=a*X$.

(2) $T_a*a=\{(a*x)*a : x\in X\}$

$=\{(a*a)*x : x\in X\}$

$=\{0*x : x\in X\}=0*X=L(X)=T_0$

(3) Let $y\in T_{a*x}$ then: $y=(a*x)*((a*x)*y)$

$=(a*x)*((a*y)*x)$

But by the proposition (1): $\leq a*(a*y)$

$\leq y$

Therefore we have: $a*(a*y)=y \Rightarrow y\in T_a$.

(4) By putting *x*=*a* in the preceding property we have:

$$T_{a*a}\subseteq T_a \Rightarrow T_0\subseteq T_a$$

(5) (i) Suppose that $z\in T_a \Rightarrow \exists x\in X ; z=a*x$.

Let $y\in X$ then by (3):

$T_{z*y}=T_{(a*x)*y}\subseteq T_{a*x}\subseteq T_a$.

So $z*y\in T_a$ and $T_a*X\subseteq T_a$

(ii) Now $T_a=a*X \subseteq T_a*X$.

(6) $x\in T_a \Rightarrow \exists u\in X ;\ x=a*u$.

We have by (3): $T_{a*u}\subseteq T_a$, therefore $T_x\subseteq T_a$.

(7) (i) *a* is an atom $\Rightarrow a\in L(X)=T_0 \Rightarrow T_a\subseteq T_0$.

And by (4) $T_0\subseteq T_a$, hence we have: $T_0=T_a$

(ii) If $T_0 = T_a$ then $a \in T_0 \Rightarrow 0*(0*a)=a$, so by lemma (2) $a \in L(X)$.

Definition (7): Let *X* be a *BCI*- algebra, then for any element $a \in X$ the subset S_a is defined by:

$$S_a = \{x \in X : x*(x*a)=a\}.$$

Proposition (3): Let *X* be a *BCI*- algebra, then for any elements $a,b \in X$ we have the following Proprieties:

(1) $a \in S_a$

(2) $x \in S_a \Rightarrow S_x \subseteq S_a$

(3) $S_a \subseteq S_b \Leftrightarrow T_b \subseteq T_a$

(4) $S_a \subseteq S_{a*x}$ for any $x \in X$

(5) $S_a = X$, if a is an atom in *X*.

(6) $S_0 = X$.

(7) If *a* is not an atom then $(X \setminus S_a)*X = X \setminus S_a$

Proof:

(1) We have $a*(a*a)=a*0=a \Rightarrow a \in S_a$

(2) Let $x \in S_a$ and let $y \in S_x$ then $x*(x*a)=a$ and $y*(y*x)=x$ which imply by proposition (1)

$$a=(y*(y*x))*((y*(y*x))*a)$$
$$=(y*(y*x))*((y*a)*(y*x))$$
$$\leq y*(y*a)$$

So $a \leq y*(y*a)$ and clearly $y*(y*a) \leq a$

So $y*(y*a)=a \Rightarrow y \in S_a$.

(3) Suppose that $S_a \subseteq S_b$ and $x \in T_b$ then $b*(b*x)=x \Rightarrow b \in S_x \Rightarrow S_b \subseteq S_x \Rightarrow S_a \subseteq S_x$, So $a \in S_x \Rightarrow x \in T_a$ Concluding $T_b \subseteq T_a$

In a similar way we can prove that $T_b \subseteq T_a \Rightarrow S_a \subseteq S_b$

(4) We know that $T_{a*x} \subseteq T_a$ therefore $S_a \subseteq S_{a*x}$.

(5) We know that $x*(x*a)=a\ (\forall x \in X)$ if *a* is an atom by lemma (1). So, $S_a = X$ if *a* is an atom.

(6) We have $0=0*0 \in 0*X \Rightarrow 0$ is an atom $\Rightarrow S_0 = X$.

(7) If a is not an atom then $S_a \neq X$.

Now let $x \in X \setminus S_a$ and $y \in X$, and suppose that $x*y \in S_a$

Then $S_{x*y} \subseteq S_a$, but $S_x \subseteq S_{x*y} \Rightarrow x \in S_a$ which is not true, so, $x*y \in X \setminus S_a$ which implies $(X \setminus S_a)*X \subseteq X \setminus S_a$, but of course:

$(X \setminus S_a)*X \supseteq X \setminus S_a$ Because for any $x \in X \setminus S_a$ we have: $x = x*0$ and finally $(X \setminus S_a)*X = X \setminus S_a$.

As a consequence of the preceding proposition we can write the following:

Corollary (1): In any *BCI* -algebra *X*, the following properties are equivalents:

(1) $S_a = S_b$

(2) $T_a = T_b$

(3) $b \in S_a \cap T_a$

(4) $a \in S_b \cap T_b$

Notation: We define a relation ~ on *BCI* - algebra *X* by:

$$x \sim y \Leftrightarrow T_x = T_y.$$

It is clear that $\sim$ is an equivalence relation, and by corollary (1) the equivalence class of an element $a \in X$ is $S_a \cap T_a$.

Proposition (4): In any *BCI* - algebra *X* the function $j_a : S_a \cap T_a \to S_0 \cap T_0;\ x \to a * x$ is well defined and injective, $\forall a \in X$.

Proof:

If $x \in S_a \cap T_a$ then: $j_a(x) = a * x$

$$= (x * (x * a)) * x$$
$$= (x * x) * (x * a)$$
$$= 0 * (x * a)$$

So $j_a(x) \in 0 * X = T_0 = X \cap T_0 = S_0 \cap T_0$

Therefore j_a is well defined.

If $x_1, x_2 \in S_a \cap T_a$ such that $j_a(x_1) = j_a(x_2)$ then: $a * x_1 = a * x_2$

Now $x_1 = a * (a * x_1) = a * (a * x_2) = x_2$

Therefore j_a is injective.

Theorem (2): In any finite *BCI*- algebra *X* the following conditions are equivalents:

(1) $(a * e) * (0 * e) = a$ $(\forall a \in X$ and $\forall e \in L(X))$.

(2) $j_a : S_a \cap T_a \to S_0 \cap T_0;\ x \to a * x$ is surjective , $\forall a \in X$.

(3) Card (0**X*) divides Card (*I*) $(\forall I \subseteq X : I * X \subseteq I)$

Proof:

(1)$\Rightarrow$(2)

By proposition (4) the function j_a is well defined and injective.

Also j_a is surjective.

For $e \in S_0 \cap T_0 = X \cap T_0 = T_0 = 0 * X = L(X)$

$\Rightarrow e$ Is an atom $\Rightarrow a*(a*e)=e.$

Let $x=a*e,$ then clearly $x\in T_a$, and $x\in S_a$ because

$$x*(x*a)=(a*e)*((a*e)*a)$$
$$=(a*e)*((a*a)*e)$$
$$=(a*e)*(0*e)$$
$$=a$$

Finally $x\in S_a\cap T_a$.

So, for $e\in S_0\cap T_0$ we have found $x\in S_a\cap T_a$ such that $j_a(x)=e$ so j_a is surjective, which prove (2).

(2)⇒(1)

Let be $a\in X$, $e\in L(X)=S_0\cap T_0$, since j_a is surjective there exists some $x\in S_a\cap T_a$ such that $j_a(x)=e$

So, $a*x=e\Rightarrow a*(a*x)=a*e$.

But $x\in T_a$ so $x = a * e$, also $x\in S_a$ implies:

$$a=(a*e)*((a*e)*a)$$
$$=(a*e)*((a*a)*e)$$
$$=(a*e)*(0*e)$$

Which prove (1).

(2) ⇒ (3)

let *I* be any subset of *X* such that $I*X\subseteq I,$ in this case it is easy to see that *I* is the union of disjoint subsets of the form $S_a\cap T_a$, because if $a\in I$ then:

$T_a=a*X\subseteq I*X\subseteq I$, but $S_a\cap T_a\subseteq T_a$ which implies that $S_a\cap T_a\subseteq I$.

so $I=\cup_{a\in C}(S_a\cap T_a)$ where $C\subseteq I$, and the subsets $S_a\cap T_a; a\in C$ are disjoints, this is possible because they are equivalence classes of the equivalence relation ~ defined above, so we have: $\text{Card}(I) = \sum_{a \in C} \text{Card}(S_a \cap T_a)$

Since j_a is surjective by (2) and injective by proposition (4) we have: $\text{Card}(S_a \cap T_a) = \text{Card}(S_0 \cap T_0)$ and:

$$\begin{aligned} \text{Card}(I) &= \sum_{a \in C} \text{Card}(S_0 \cap T_0) \\ &= \sum_{a \in C} \text{Card}(0 * X) \\ &= \text{Card}C \times \text{Card}(0 * X) \end{aligned}$$

Which implies that: Card (0 * X) divides Card (I).

(3) $\Rightarrow$ (2):

If Card (0 * X) divides Card (I) $(\forall I \subseteq X : I * X \subseteq I)$ then:

For any $a \in X$ we have: $T_a = X \cap T_a$

$$\begin{aligned} &= (S_a \cup (X \setminus S_a)) \cap T_a \\ &= (S_a \cap T_a) \cup ((X \setminus S_a) \cap T_a) \end{aligned}$$

This implies $\text{Card}T_a = \text{Card}(S_a \cap T_a) + \text{Card}((X \setminus S_a) \cap T_a)$

Or $\text{Card}(S_a \cap T_a) = \text{Card}T_a - \text{Card}((X \setminus S_a) \cap T_a)$

We have tow cases:

(i)If a is not an atom then: $(X \setminus S_a) * X = (X \setminus S_a)$ (Proposition 3)

And $T_a * X = T_a$ (Proposition 2)

Hence $((X \setminus S_a) \cap T_a) * X \subseteq (X \setminus S_a) \cap T_a$ By (3) we have:

Card (0 * X) $\mid \text{Card}(T_a)$, Card (0 * X) $\mid \text{Card}((X \setminus S_a) \cap T_a)$

This implies:

Card(0*X) $\mid [\text{Card}(T_a) - \text{Card}((X \setminus S_a) \cap T_a)] \Longrightarrow$

Card (0* X) $\mid \text{Card}(S_a \cap T_a)$

However, clearly $S_0 \cap T_0 = 0 * X$, which implies:

$\text{Card}(S_0 \cap T_0) \mid \text{Card}(S_a \cap T_a) \Longrightarrow \text{Card}(S_0 \cap T_0) \leq \text{Card}(S_a \cap T_a)$

By proposition (4) the function j_a is injective, so we got the inequality: $\mathrm{Card}(S_a \cap T_a) \le \mathrm{Card}(S_0 \cap T_0)$ concluding that:

$\mathrm{Card}(S_a \cap T_a) = \mathrm{Card}(S_0 \cap T_0) \Rightarrow j_a$ is surjective, because it is an injective function of finite subsets which have the same cardinality.

(ii) If a is an atom, we know that $S_a = X = S_0$ by Proposition (3), $T_a = T_0$ by Proposition (2).

Therefore, we have: $S_a \cap T_a = S_0 \cap T_0 \Rightarrow$ similarly, as in (i), j_a is surjective.

So j_a is surjective $\forall a \in X$, which prove (2).

Which conclude the proof of the theorem.

By the preceding theorem and by theorem (1) we have:

Corollary (1): A finite *BCI* - algebra *X* is of *KL*- product iff: Card (0 * *X*) divides Card (*I*) $(\forall I \subseteq X : I * X \subseteq I)$.

Remark: let *I* be a subset of finite *BCI*- algebra *X* such that $I * X \subseteq I$ then we can see that if $x \in I$ then $T_x = x * X \subseteq I$, now using the fact that $x \in T_x$ by (2) in lemma (3) we can write: $I = \bigcup_{x \in I} T_x$ Or $I = T_{a_1} \cup T_{a_2} \cup \ldots.. \cup T_{a_s}$ in such a way that $T_{a_i} \not\subset T_{a_j}$ whenever $i \neq j$, in this case we shall say that *I* is properly written.

Theorem (3): Let be *X* a finite *BCI* - algebra, then the following conditions are equivalents:

(1) Card (0 * *X*) divides Card (*I*) $(\forall I \subseteq X : I * X \subseteq I)$

(2) Card (0 * *X*) divides Card (0 * *X*) $(\forall a \in X)$

Proof:

(1) $\Rightarrow$ (2) Let $I = T_a$ where $a \in X$.

By the property (5) of the proposition (2) the subset I satisfies the condition $I * X \subseteq I$, so, by (1) Card (0 * X) divides Card (I), but $I = T_a = a * X$, which is clear by the proposition (2), and condition (2) is proved.

(2) $\Rightarrow$ (1)

First we define the subsets: $X_n = \{x \in X : \text{Card}(T_x) = n\}$, for n any natural number, then we have a sequence of natural numbers: $n_0 < n_1 < n_2 < .... < n_k < ....$ where we suppose that for any number n_k there exist at least one element $x \in X$ such that: $\text{Card}(T_x) = n_k$ and if we have $n_k < m < n_{k+1}$ then there is no element $x \in X$ such that $\text{Card}(T_x) = m$.

It is clear by propriety (4) of the proposition (2) that $\text{Card}(T_0) = n_0$.

We will prove (1) by induction on k the index of the numbers n_k.

Now, the statement of induction for $k-1$ is:

If I is any subset of Xsuch that:

$I * X \subseteq I$ and $\text{Card}(T_x) \le n_{k-1}; \forall x \in I$ then:

Card (0 * X) divides Card (I).

Now for $k = 0$, Suppose that I is a subset of X such that $I * X \subseteq I$ and $\text{Card}(T_x) \le n_0; \forall x \in I$, but $T_0 \subseteq T_x$ by proposition (2), so $\text{Card}(T_0) \le \text{Card}(T_x) \le n_0$ then it is clear that $\text{card}(T_x) = n_0$, because the fact mentioned above that $\text{Card}(T_0) = n_0$, also we have: $T_0 = T_x$, $\forall x \in I$ however $I = \bigcup_{x \in I} T_x$ therefore, in this case $I = T_0$.

So, $\text{Card}(T_0)$ divides Card (I), or Card (0 * X) divides Card (I), and the statement of induction is true for $k=0$.

Now, we prove the statement for k:

Suppose that I is any subset of Xsuch that $I * X \subseteq I$ and

$\mathrm{Card}(T_x) \le n_k ; \forall x \in I$, We shall prove that Card $(0 * X)$ divides Card (I), provided that the assertion is true for k-1.

As we have seen above in the remark, the subset I can be properly Written in the form: $I = T_{a_1} \cup T_{a_2} \cup ..... \cup T_{a_s}$.

By set theory we have:

$$\mathrm{Card}(I) = \sum_i \mathrm{Card}(T_{a_i}) - \sum_{i<j} \mathrm{Card}(T_{a_i} \cap T_{a_j}) + \sum_{i<j<k} \mathrm{Card}(T_{a_i} \cap T_{a_j} \cap T_{a_k}) - ...$$

Now it is clear that the first sum is divisible by $\mathrm{Card}(0 * X)$, because this number divides $\mathrm{Card}(T_{a_i}), \forall i$ by (2).

For the other sums, the subsets like $T_{a_i} \cap T_{a_j}$, $T_{a_i} \cap T_{a_j} \cap T_{a_k}$, …… will be denoted by J.

We shall prove the following for the subset $J = T_{a_i} \cap T_{a_j}$, which we can do with the remaining subsets in the same way.

It is clear by property (5) in the proposition (2) that the subset J Satisfies the condition: $J * X \subseteq J$ and can be written as the union of subsets of the form T_x, just as it was remarked above so: $J = T_{a_i} \cap T_{a_j} = T_{b_1} \cup T_{b_2} \cup ..... \cup T_{b_r}$, (Properly Written), here we have: $\mathrm{Card}(T_{b_t}) \le n_{(k-1)}, (1 \le t \le r)$, because if $\mathrm{Card}(T_{b_t}) = n_k$, for some $1 \le t \le r$ then we have:

$$n_k = \mathrm{Card}(T_{b_t}) \le \mathrm{Card}(T_{a_i} \cap T_{a_j}) \le \mathrm{Card}(T_{a_i}) \le n_k$$

$$n_k = \mathrm{Card}(T_{b_t}) \le \mathrm{Card}(T_{a_i} \cap T_{a_j}) \le \mathrm{Card}(T_{a_j}) \le n_k$$

So, we have: $\mathrm{Card}(T_{a_i} \cap T_{a_j}) = \mathrm{Card}(T_{a_i}) = \mathrm{Card}(T_{a_j})$ which implies $T_{a_i} = T_{a_i} \cap T_{a_j} = T_{a_j}$ but this is impossible because I is properly Written in the form $I = T_{a_1} \cup T_{a_2} \cup ..... \cup T_{a_s}$, so

$\text{Card}(T_{b_t}) \le n_{(k-1)}, (1 \le t \le r)$, from this we have by using property (6) in proposition (2) that:

$\text{Card}(T_x) \le n_{(k-1)}; \forall x \in J$ And by the statement of induction:

$\text{Card}(0 * X)$ divides $\text{Card}(J) = \text{Card}(T_{a_i} \cap T_{a_j})$.

In the same way we proceed for the remaining subsets in the Other sums, so Card (0 * *X*) divides each of these sums, hence: Card (0 * *X*) divides Card (*I*). Which prove (1).

So the theorem is proved.

By the preceding theorem and corollary (1) of the theorem (2) we have:

Corollary (1):

A finite *BCI*- algebra *X* is of *KL*- product iff Card (0 * *X*) divides $\text{Card}(a * X) \; (\forall a \in X)$.

In other words a finite *BCI* -algebra *X* is of *KL*- product iff the number of atoms in *X* divides the number of elements of any row in the Cayley table of the binary operation*.

Example:

The last corollary gives us a simple method for examining if a *BCI* -algebra is of *KL*- product or not. Here we have these tow examples:

*	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	0	4	3	3
2	2	2	0	5	5	3
3	3	3	3	0	0	0
4	4	3	3	1	0	0
5	5	5	3	2	2	0

*	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	0	3	3	3
2	2	2	0	5	5	3
3	3	3	3	0	0	0
4	4	3	3	1	0	0
5	5	5	3	2	2	0

Where the first table represents a *BCI*- algebra of *KL*- product but the second does not. Because the number of atoms in the first and the second *BCI*-algebras is 2 (the number of the elements of the first row in the tow tables), it is clear that the condition of the precedent corollary is satisfied in the first table but not in the second.

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