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Locally Projective and Locally Injective Modules

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ABSTRACT

The object of this paper is to study the endomorphism rings of locally projective and locally injective modules. Specifically, this paper is a continuation of study of endomorphism rings of locally projective and locally injective modules to be semipotent rings. The main obtained results include:

(a) ***Let*** P_R ***be a locally projective module over a ring*** R ***, then for any*** $M \in mod - R$ ***the following are equivalent:***

(1) $[M, P]$ ***is a semipotent.***

(2) $\text{Tot}[M, P] = J[M, P] = \nabla [M, P]$.

(3) ***For any*** $\alpha \in [M,P] \setminus J[M,P]$ ***there exists*** $\beta \in [P,M]$ ***with*** $0 \neq Im(\alpha\beta) \subseteq^{\oplus} P$.

In particular, the endomorphisms ring E_P ***of*** P ***is a semipotent ring if and only if, for any*** $\alpha \in E_P \setminus J(E_P)$ ***there exists*** $0 \neq \beta \in E_P$ ***such that*** $o \neq Im(\alpha\beta) \subseteq^{\oplus} P$.

(b) ***Let*** Q_R ***be a locally injective module over a ring*** R ***, then for any module*** $N \in mod - R$ ***the following are equivalent:***

(1) $[Q, N]$ ***is a semipotent.***

(2) $\text{Tot}[Q, N] = J[Q, N] = \Delta[Q, N]$.

(3) ***For any*** $\alpha \in [Q, N] \setminus J[Q, N]$ ***there exists*** $\beta \in [N,Q]$ ***with*** $0 \neq Ker(\beta\alpha) \subseteq^{\oplus} Q$.

In particular, E_Q ***is a semipotent ring if and only if, for any*** $\alpha \in E_Q \setminus J(E_Q)$ ***there exists*** $0 \neq \beta \in E_Q$ ***such that*** $0 \neq Ker(\beta\alpha) \subseteq^{\oplus} Q$.

Key Words: *Semipotent Rings, Locally projective and locally injective modules, The total, Jacobson radical, (co) singular ideal, Endomorphisms rings,* $hom_R(M, N)$.

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المودولات الإسقاطية المحلية والأفقية المحلية

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الملخص

الهدف من هذا البحث هو دراسة حلقات التشاكل للمودولات الإسقاطية المحليـة والأفقيـة المحليـة. بشكل خاص، تعد هذه الورقة متابعة لدراسة حلقات التشاكل للمودولات الإسقاطية والأفقية المحلية لكـي تكون هذه الحلقات شبه جامدة. النتائج الرئيسية التي تم التوصل إليها من خلال هذه الدراسة هي:

(أ) ليكن P_R مودولاً إسقاطياً محلياً فوق الحلقة R عندئذٍ لأجل أي مـودول $M \in mod - R$ الشروط الآتية متكافئة:

1. المودول الثنائي $[M, P]$ شبه جامد.

2. $\text{Tot}\,[M,P] = J\,[M,P] = \nabla[M,P]$.

3. أياً كان $\alpha \in [M\ ,P] \setminus J\,[M\ ,P]$ يوجد $\beta \in [P,M]$ يحقق $0 \neq Im\ (\alpha\beta) \subseteq^{\oplus} P$.

بشكل خاص، الشرط اللازم والكافي كي تكون حلقة التشاكلات E_P لمودول إسقاطي محلي P_R حلقة شبه جامدة هو تحقق الشرط: أياً كان $\alpha \in E_P \setminus J(E_P)$ يوجد $0 \neq \beta \in E_P$ يحقق $0 \neq Im\,(\alpha\beta) \subseteq^{\oplus} P$.

(ب) ليكن Q_R مودولاً أفقياً محلياً فوق الحلقة R عندئذٍ لأجـل أي مـودول $N \in mod - R$ الشروط الآتية متكافئة:

1. المودول الثنائي $[Q, N]$ شبه جامد.

2. $\text{Tot}\,[Q,N] = J\,[Q,N] = \Delta[Q,N]$.

3. أياً كان $\alpha \in [Q,N] \setminus J\,[Q,N]$ يوجد $\beta \in [N\ ,Q]$ يحقق $0 \neq Ker\ (\beta\alpha) \subseteq^{\oplus} Q$.

بشكل خاص، الشرط اللازم والكافي كي تكون حلقة التشاكلات E_Q لمودول أفقي محلي Q_R حلقة شبه جامدة هو تحقق الشرط: أياً كان $\alpha \in E_Q \setminus J(E_Q)$ يوجد $0 \neq \beta \in E_Q$ يحقق $0 \neq Ker\,(\beta\alpha) \subseteq^{\oplus} Q$.

الكلمات المفتاحية: الحلقات شبه الجامدة، المودولات الإسقاطية المحلية، المـودولات الأفقية المحلية، التوتال، أساس جاكبسون، المثالي المنفرد، حلقـة التشاكلات لمودول، $hom_R(M, N)$.

رقم التصنيف العالمي: $16E50, 16D40$.

1. Introduction.

In this paper rings R are associative with identity unless otherwise indicated. All modules over a ring R are unitary right modules. A submodule N of a module M is said to be small in M if $N+K \neq M$ for any proper submodule K of M, [3]. A submodule N of a module M is said to be large (essential) in M if $N \cap K \neq 0$ for any nonzero submodule K of M, [3]. If M is an $R-$module, the radical of M denoted by $J(M)$ is defined to be the intersection of all maximal submodules of M. Also, $J(M)$ coincides with the sum of all small submodules of M. It my happen that M has no maximal submodules in which case $J(M)=M$, [8]. Thus, for a ring R, $J(R)$ is the Jacobson radical of R. Also, we write $U(R)$ the group of units of a ring R. For a submodule N of a module M, we use $N \subseteq^{\oplus} M$ to mean that N is a direct summand of M, and we write $N \leq_e M$ and $N << M$ to indicate that N is a large, respectively small, submodule of M. If M_R is a module, we use the notation $E_M = End_R(M)$ and we write $\Delta E_M = \{\alpha : \alpha \in E_M ; Ker(\alpha) \leq_e M\}$, $\nabla E_M = \{\alpha : \alpha \in E_M ; Im(\alpha) << M\}$ and $I(E_M) = \{\alpha : \alpha \in E_M ; Im(\alpha) \subseteq J(M)\}$. It is will known that ΔE_M, ∇E_M and $I(E_M)$ are ideals in E_M [3]. If M_R and N_R are modules, we use $[M,N]= hom_R(M,N)$. Thus, $[M,N]$ is an $(E_M,E_N)-$bimodule. Our main concern is about the substructures of $hom_R(M,N)$ and the semipotent of $hom_R(M,N)$ (see [9]).

In this paper we study the following two questions (see [3, p. 1504]). (1) *If P is locally projective module, when is it true that* $\text{Tot}[M,P]=\nabla[M,P]=J[M,P]$ *for all $M \in mod-R$?.* (2) *If Q is locally injective module, when is it true that* $\text{Tot}[Q,N]=\Delta[Q,N]=J[Q,N]$ *for all $N \in mod-R$?.*

In section (2), *it is proved that if, P is a locally projective module then $[M,P]$ is an $I-$semipotent for all $M \in mod-R$. The main result in this section, if P is locally projective then for all $M \in mod-R$,* $\text{Tot}[M,P]=\nabla[M,P]=J[M,P]$ *if and only if,*

$[M,P]$ *is semipotent which also, equivalent, for any* $\alpha \in [M,P] \setminus J[M,P]$ *there exists* $\beta \in [P,M]$ *with* $0 \neq Im(\alpha\beta) \subseteq^{\oplus} P$. *In section* (3), *it is proved that if,* Q *is a locally injective module then* $[Q,N]$ *is an* $I-$*semipotent for all* $N \in mod-R$. *The main result in this section, if* Q *is locally injective then for all* $N \in mod-R$, $\mathrm{Tot}[Q,N] = \Delta[Q,N] = J[Q,N]$ *if and only if,* $[Q,N]$ *is semipotent which also, equivalent, for any* $\alpha \in [Q,N] \setminus J[Q,N]$ *there exists* $\beta \in [N,Q]$ *with* $0 \neq Ker(\beta\alpha) \subseteq^{\oplus} Q$.

Following [9], let M_R, N_R *be modules. We put* $[M,N] = hom_R(M,N)$. *Thus,* $[M,N]$ *is an* (E_M, E_N)*-bimodule. The four substructures of* $hom_R(M,N)$ *given as follows (see [9]).*

- *The Jacobson radical*

$J[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in J(E_M)$ *for all* $\beta \in [N,M]\}$

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$J[M,N] = \{\alpha : \alpha \in [M,N];\ 1_N - \alpha\beta \in U(E_N)$ *for all* $\beta \in [N,M]\}$

$J[M,N] = \{\alpha : \alpha \in [M,N];\ 1_M - \beta\alpha \in U(E_M)$ *for all* $\beta \in [N,M]\}$

Thus, $J[M,M] = J(E_M)$. *In particular,* $J[R,R] = J(R)$.

- *The singular ideal*

$$\Delta[M,N] = \{\alpha : \alpha \in [M,N];\ Ker(\alpha) \leq_e M\}.$$

- *The co-singular ideal*

$$\nabla[M,N] = \{\alpha : \alpha \in [M,N];\ Im(\alpha) << N\}.$$

- *The total*

$\mathrm{Tot}[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha\,[N,M]$ *contains no nonzero idempotents}*

$\mathrm{Tot}[M,N] = \{\alpha : \alpha \in [M,N];\ [N,M]\alpha$ *contains no nonzero idempotents}*

Thus,

$\mathrm{Tot}[M,M] = \mathrm{Tot}(E_M) = \{\alpha : \alpha \in E_M;\ \alpha E_M$ *contains no nonzero idempotents}*

$= \{\alpha : \alpha \in E_M;\ E_M\,\alpha$ *contains no nonzero idempotents}*.

- *Also, we put*

$I[M,N]=\{\alpha:\alpha\in[M,N];\, Im(\alpha)\subseteq J(N)\}$

It is clear that

$I[M,N]\subseteq\{\alpha:\alpha\in[M,N];\, \beta\alpha\in I(E_M)$ *for all* $\beta\in[N,M]\}$

$I[M,N]\subseteq\{\alpha:\alpha\in[M,N];\, \alpha\beta\in I(E_N)$ *for all* $\beta\in[N,M]\}$.

Since any small submodule of N *contained in* $J(N)$ *then* $\nabla[M,N]\subseteq I[M,N]$. *If* $J(N)<<N$ *then* $\nabla[M,N]=I[M,N]$. *Thus* $I=I(E_M)=I[M,M]=\{\alpha:\alpha\in E_M;\ \ Im(\alpha)\subseteq J(M)\}$. *In particular for a ring* R, $I(R)=I[R,R]=J[R,R]=J(R)$.

2. Locally Projective Modules.

Recall a projective module P_R *is an* I_0-*module [1] if, any submodule* A *of* P, $A\not\subset J(P)$ *contains a nonzero direct summand of* P. *A ring* R *is called semipotent ring also, called* I_0-*ring [1,5] if, every principal leftl (resp. right) ideal not contained in* $J(R)$ *contains a nonzero idempotent. A module* P_R *is called locally projective [4] if, for every submodule* $B\subseteq P$, *which is not small in* P *there exists a projective direct summand* $0\neq W\subseteq^{\oplus}P$ *with* $W\subseteq B$. *Thus, every projective* I_0- *module* P *with* $J(P)<<P$ *is locally projective or equivalently, any projective module* P *with* E_P *is semipotent, is locally projective. For any module* M_R, *we have* $J(M)=\{a:a\in M;aR<<M\}$.

Lemma 2.1. *For any locally projective module* P *the following hold:*

(1) $J(P)<<P$.

(2) *A submodule* K *of* P *is small in* P *if and only if* $K\subseteq J(P)$.

(3) $J(E_P)\subseteq\nabla\, E_P$.

Proof. (1). *Suppose that* $J(P)$ *is not small in* P *then* $J(P)$ *contains a nonzero projective direct summand submodule* D *of* P.

Thus $J(D)=D$ *which contradicts that* $D\neq 0$ *projective, therefore* $J(P)<<P$.

(2) $(\Rightarrow)$. *It is clear that if* $K<<P$ *then* $K\subseteq J(P)$. $(\Leftarrow)$. *Let* $K\subseteq J(P)$. *Suppose that* K *is not small in* P *then there exists a projective submodule* N *of* P, *which* $0\neq N\subseteq^{\oplus} P$ *and* $N\subseteq K$. *Since* $N\subseteq^{\oplus} P$ *and* $N\subseteq J(P)$ *follow* $J(N)=N\cap J(P) \ =N$ *which contradicts that* $N\neq 0$ *projective, thus* $K<<P$.

(3). *Let* $\alpha\in J(E_P)$. *Suppose* $Im(\alpha)\not\subset J(P)$ *by (2),* $Im(\alpha)$ *not small in* P, *so there exists a projective direct summand submodule* $0\neq W\subseteq^{\oplus} P$ *with* $W\subseteq Im(\alpha)$. *Let* $e:P\rightarrow W$ *the projection of* P *onto* W *then* $0\neq e^2=e\in E_P$ *and* $W=Im(e)=\ Im(e\alpha)$. *Since* W *is projective then* $Ker(e\alpha)\subseteq^{\oplus} P$. *Thus,* $Im(e\alpha)$ *and* $Ker(e\alpha)$ *are direct summands of* P, *by [8, lemma 3.1],* $e\alpha=(e\alpha)\varphi(e\alpha)$ *for some* $0\neq\varphi\in E_P$, *therefore* $0\neq\varphi(e\alpha)\in E_P$ *is an idempotent and* $\varphi(e\alpha)\in J(E_P)$ *which contradicts that* $J(E_P)$ *doesn't contains a nonzero idempotent, thus* $Im(\alpha)\subseteq J(P)$ *by (2) follows that* $J(E_P)\subseteq\nabla\ E_P$.

Corollary 2.2. *A module* P *with* $J(P)<<P$ *is locally projective if and only if every submodule* K *of* P, $K\not\subset J(P)$ *contains a nonzero projective direct summand of* P.

Proof follows immediately from lemma 2.1.

Following [10], a module M *is called a regular module if given any* $m\in M$ *there exists* $f\in[M,R]$ *with* $m=mf(m)$, *or equivalently, for any* $m\in M$, mR *is a projective direct summand of* M *(see [10, Theorem 2.2]). Therefore any regular module* M *is locally projective with* $J(M)=0$. *It is known that if* M *is a regular module then* E_M *is a semipotent ring with* $J(E_M)=0$, *(see [1, Corollary 3.6]).*

Proposition 2.3. *Endomorphism ring of every locally projective module* P *with* $J(P)=0$ *is a semipotent ring and* $J(E_P)=0$.

Proof. Let P *be a locally projective module with* $J(P)=0$ *by lemma 2.1(3), follows that* $J(E_P)=0$. *Let* $0\neq\alpha\in E_P$ *then* $Im(\alpha)$ *is not small submodule in* P, *since* P *is locally projective there exists a projective direct summand* $W\neq 0$ *of* P *which* $W\subseteq Im(\alpha)$. *Let* $\gamma:P\to W$ *be the projection of* P *onto* W *then* $0\neq\gamma=\gamma^2\in E_P$ *and* $Im(\gamma)=Im(\gamma\alpha)=W$. *Since* W *is projective then* $Ker(\gamma\alpha)\subseteq^{\oplus} P$ *by [8, lemma 3.1] there exists* $\mu\in E_P$ *such that* $(\gamma\alpha)\mu(\gamma\alpha)=\gamma\alpha$. *We put* $\beta=\mu\gamma\alpha\mu\gamma$ *then* $0\neq\beta\in E_P$ *and* $\beta=\beta\alpha\beta$, *thus* E_P *is semipotent.*

Theorem 2.4. *Let* P_R *be a locally projective module. The following are equivalent:*

(1) E_P *is a semipotent ring.*

(2) $J(E_P)=\nabla E_P$.

(3) *For every* $\alpha\in E_P\setminus J(E_P)$, $Im(\alpha)$ *contains a projective direct summand submodule* $0\neq B\subseteq P$.

Proof. $(1)\Rightarrow(2)$. *We have by lemma 2.1(3),* $J(E_P)\subseteq\nabla E_P$. *Suppose that* $\nabla E_P\not\subset J(E_P)$. *Then there exists* $\alpha\in\nabla E_P$, $\alpha\notin J(E_P)$, *thus* $0\neq\alpha\in E_P$ *and* $Im(\alpha)<<P$. *Since* E_P *is semipotent then* $\beta=\beta\alpha\beta$ *for some* $0\neq\beta\in E_P$ *therefore* $Im(\beta\alpha)\subseteq^{\oplus} P$ *and* $Im(\beta\alpha)<<P$. *Since* $P=Im(\beta\alpha)\oplus K=K$ *for some submodule* $K\neq 0$ *of* P, *follows* $Im(\beta\alpha)\subseteq Im(\beta\alpha)\cap K=0$ *therefore* $\beta\alpha=0$ *and* $\beta=0$, *a contradiction, thus* $J(E_P)=\nabla E_P$.

$(2)\Rightarrow(3)$. *Let* $\alpha\in E_P\setminus J(E_P)$, *then* $\alpha\notin\nabla E_P$ *thus* $Im(\alpha)$ *is not small in* P *therefore* $Im(\alpha)$ *contains a projective direct summand submodule* $B\neq 0$ *of* P. *Suppose* (3) *holds. Let* $g\in E_P$, $g\notin J(E_P)$ *then* $Im(g)$ *contains a projective direct summand submodule* $D\neq 0$ *of* P. *Let* $\gamma:P\to D$ *the projection of* P *onto* D.

Since $Im(\gamma)=D\subseteq Im(g)$ *follows that* $D=Im(\gamma)=Im(\gamma g)$, *hence* $\gamma\neq 0$ *is an idempotent of* E_P, *since* $Im(\gamma)=D$ *is projective then* $Ker(\gamma g)\subseteq^{\oplus} P$ *by [1, Theorem 2.2] follows that* E_P *is a semipotent ring. This proves* $(3)\Rightarrow(1)$.

Corollary 2.5. *A projective module* P_R *is locally projective if and only if* E_P *is a semipotent ring. In particular,* R_R *(resp.* $_R R$*) is a locally projective module if and only if,* R *is a semipotent ring.*

Proof. Let P *be a projective module.* $(\Rightarrow)$. *If* P *is a locally projective then by lemma 2.1,* $J(P)<<P$ *and by [1, Theorem 3.5]* E_P *is a semipotent ring.* $(\Leftarrow)$. *Suppose that* E_P *is semipotent again by [1, Theorem 3.5]* P *is locally projective.*

Lemma 2.6. *Let* P_R *be a locally projective module, then for any module* $M\in$ $mod-R$, *the following hold:*

(1) $\mathrm{Tot}[M,P]=\nabla[M,P]=I[M,P]$.

(2) $J[M,P]\subseteq I[M,P]$.

(3) $\Delta[M,P]\subseteq I[M,P]$.

(4) $\Delta[M,P]\subseteq\nabla[M,P]$.

In particular, $\Delta E_P\subseteq I(E_P)=\nabla E_P=\mathrm{Tot}(E_P)$.

Proof. (1). *Kasch in [5], proved that* $\mathrm{Tot}[M,P]=\nabla[M,P]$ *for any module* $M\in$ $mod-R$. *Since by lemma 2.1,* $J(P)<<P$ *follows* $I[M,P]=\nabla[M,P]$.

(2). *Let* $\alpha\in J[M,P]$. *Suppose* $\alpha\notin I[M,P]$ *then by (2) there exists* $\beta\in[P,M]$ *such that* $0\neq\alpha\beta=(\alpha\beta)^2\in E_P$. *Since* $\alpha\in J[M,P]$ *then* $0\neq\alpha\beta\in J(E_P)$ *a contradiction.*

(3). *Let* $\alpha\in\Delta[M,P]$ *then* $Ker(\alpha)\leq_e M$. *Suppose* $\alpha\notin I[M,P]$ *then by (2) there exists* $\gamma\in[P,M]$ *such that* $0\neq\gamma\alpha=(\gamma\alpha)^2\in E_M$. *Since* $Ker(\alpha)\subseteq Ker(\gamma\alpha)$ *therefore* $Ker(\gamma\alpha)\leq_e M$ *and*

$Ker(\gamma\alpha) \cap Im(\gamma\alpha) = 0$ *thus* $Im(\gamma\alpha) = 0$, $\gamma\alpha = 0$ *this is a contradiction, so* $\Delta[M,P] \subseteq I[M,P]$. (4). *It is clear by (1) and (3).*

Lemma 2.7. *[9, Lemma 2.1]. Let* M_R, N_R *be modules. The following conditions are equivalent:*

(1) *If* $\alpha \in [M,N] \setminus J[M,N]$ *there exists* $\beta \in [N,M]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$.

(2) *If* $\alpha \in [M,N] \setminus J[M,N]$ *there exists* $\beta \in [N,M]$ *such that* $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$.

(3) *If* $\alpha \in [M,N] \setminus J[M,N]$ *there exists* $\gamma \in [N,M]$ *such that* $\gamma\alpha\gamma = \gamma \notin J[N,M]$.

Following [9], Recall that $[M,N]$ *is semipotent if, the conditions in lemma 2.7 are satisfied. Thus* $[M,M]$ *is semipotent if and only if* E_M *is a semipotent ring.*

Theorem 2.8. *Let* P_R *be a locally projective module. For any module* $M \in mod-R$ *the following are equivalent:*

(1) $[M,P]$ *is a semipotent.*

(2) $\mathrm{Tot}[M,P] = J[M,P] = \nabla[M,P]$.

(3) *For any* $\alpha \in [M,P] \setminus J[M,P]$ *there exists* $\beta \in [P,M]$ *with* $0 \neq Im(\alpha\beta) \subseteq^{\oplus} P$.

In particular, E_P *is a semipotent ring if and only if, for any* $\alpha \in E_P \setminus J(E_P)$ *there exists* $0 \neq \beta \in E_P$ *such that* $Im(\beta\alpha) \subseteq^{\oplus} P$.

Proof. $(1) \Rightarrow (2)$. *Suppose that* $[M,P]$ *is semipotent then [9, Theorem 2.2]* $\mathrm{Tot}[M,P] = J[M,P]$ *and by lemma 2.6* $J[M,P] = \nabla[M,P]$. $(2) \Rightarrow (1)$. *Since* $\mathrm{Tot}[M,P] = J[M,P]$ *then by [9, Theorem 2.2],* $[M,P]$ *is semipotent.*

$(1)\Rightarrow(3)$. *Let* $\alpha\in[M,P]\setminus J[M,P]$ *then there exists* $\beta\in[P,M]$ *such that* $0\neq\alpha\beta=(\alpha\beta)^2\in E_P$, *so* $0\neq Im(\alpha\beta)\subseteq^{\oplus}P$.

$(3)\Rightarrow(2)$. *Since* P *is locally projective then by lemma 2.6* $J[M,P]\subseteq\nabla[M,P]$. *Let* $\alpha\in\nabla[M,P]$, *suppose that* $\alpha\notin J[M,P]$ *then there exists* $\beta\in[P,M]$ *such that* $0\neq Im(\alpha\beta)\subseteq^{\oplus}P$. *Since* $\alpha\in\nabla[M,P]$ *and* $Im(\alpha\beta)\subseteq Im(\alpha)$ *then* $Im(\alpha\beta)<<P$. *Therefore* $Im(\alpha\beta)=0$ *and* $\alpha\beta=0$, *a contradiction. Thus,* $\alpha\in J[M,P]$.

The semipotent rings generalized as following:

Lemma 2.9. *[7, Lemma 19]. The following conditions are equivalent for an ideal* I *of a ring* R*:*

(1) *If* $T\not\subset I$ *is a right (resp. left) ideal there exists* $e^2=e\in T\setminus I$.

(2) *If* $a\notin I$ *there exists* $e^2=e\in aR\setminus I$ *(resp.* $e^2=e\in Ra\setminus I$ *).*

(3) *If* $a\notin I$ *there exists* $x\in R$ *such that* $x=xax\notin I$.

Let R *be a ring and* I *is an ideal of* R, *recall* R *is an* $I-$*semipotent [7], if the conditions in lemma 2.9, are satisfied.*

Corollary 2.10. *Let* I *be an ideal of a ring* R. *If* R *is* $I-$*semipotent then* $J(R)\subseteq I$.

Proof. Suppose $J(R)\not\subset I$ *there exists* $a\in J(R)$, $a\notin I$, *so* $x=xax\notin I$ *for some* $x\in R$. *Since* $x\neq 0$ *then* $0\neq(ax)^2=ax\in J(R)$ *this is a contradiction.*

Lemma 2.11. *Let* M_R, N_R *be modules. The following conditions are equivalent:*

(1) *If* $\alpha\in[M,N]\setminus I[M,N]$, *there exists* $\beta\in[N,M]$; $0\neq\beta\alpha=(\beta\alpha)^2\in E_M$, $\beta\alpha\notin I(E_M)$.

(2) *If* $\alpha\in[M,N]\setminus I[M,N]$, *there exists* $\beta\in[N,M]$; $0\neq\alpha\beta=(\alpha\beta)^2\in E_N$, $\alpha\beta\notin I(E_N)$.

(3) *If* $\alpha \in [M,N] \setminus I[M,N]$, *there exists* $\gamma \in [N,M]$; $\gamma\alpha\gamma = \gamma \notin I[N,M]$.

Proof. Suppose (1) *holds. Then* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ *and* $\beta\alpha \notin I(E_M)$ *for some* $\beta \in [N,M]$. *By letting* $\gamma = \beta\alpha\beta \in [N,M]$ *we have* $\gamma\alpha\gamma = \gamma \neq 0$ *and* $\gamma \notin I[N,M]$ *because* $\beta\alpha \notin I(E_M)$, *giving (3). Suppose(3) holds. Then* $0 \neq \gamma\alpha = (\alpha\gamma)^2 \in E_M$ *and* $\gamma\alpha \notin I(E_M)$ *because* $\gamma \notin I[N,M]$ *gives (1). Similarly, the equivalence* $(2) \Leftrightarrow (3)$ *holds.*

We call that $[M,N]$ *is* $I-$*semipotent if, the conditions in lemma 2.11 are satisfied. If* $I[M,N] = J[M,N]$ *then* $[M,N]$ *is semipotent if and only if* $[M,N]$ *is* $I-$ *semipotent.*

Proposition 2.12. *Let* P_R *be a locally projective module, then the following hold:*

(1) E_P *is an* $I-$ *semipotent ring, where* $I = I(E_P) \subseteq E_P$.

(2) $[M,P]$ *is an* $I-$ *semipotent for any module* $M \in mod-R$.

Proof. (1). *Let* $\alpha \in E_P$, $\alpha \notin I(E_P)$ *then* $Im(\alpha) \not\subset J(P)$ *and by lemma 2.1,* $Im(\alpha)$ *is not small in* P . *So, there exists a projective direct summand* $0 \neq N \subseteq^{\oplus} P$ *with* $N \subseteq Im(\alpha)$. *Let* β *be the projection of* P *on to* N , *then* $Im(\beta) \subseteq Im(\alpha)$ *and* $Im(\beta\alpha) \subseteq Im(\beta) = N \subseteq^{\oplus} P$. *Since* N *is projective then* $Ker(\beta\alpha) \subseteq^{\oplus} P$, *by [8, Lemma 3.1] there exists* $\gamma \in E_P$ *such that* $(\beta\alpha)\gamma(\beta\alpha) = \beta\alpha$. *So,* $0 \neq (\gamma\beta\alpha)^2 = \gamma\beta\alpha$

$\in E_P.\alpha$, *and* $\gamma\beta\alpha \notin I(E_P)$, *hence* $J(P) << P$. *Thus,* E_P *is an* $I-$ *semipotent ring.*

(2). *Let* $\alpha \in [M,P] \setminus I[M,P]$ *then by lemma 2.6* $\alpha \notin \mathrm{Tot}[M,P]$ *so there exists* $\beta \in [P,M]$ *such that* $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_P$ *and* $\alpha\beta \notin I(E_P)$, *hence* $J(P) << P$ *thus by lemma 2.11,* $[M,P]$ *is* $I-$*semipotent.*

3. Locally injective Modules.

Recall a module Q_R *is a locally injective [4] if, for every submodule* $A \subseteq Q$ *which is not large in* Q *there exists an injective submodule* $0 \neq B \subseteq Q$ *with* $A \cap B = 0$.

Lemma 3.1. *Let* Q_R *be a locally injective module. Then for any module* $N \in mod - R$ *the following hold:*

(1) $\mathrm{Tot}[Q, N] = \Delta[Q, N]$.

(2) $J[Q, N] \subseteq \Delta[Q, N]$.

(3) $\nabla[Q, N] \subseteq \Delta[Q, N]$.

In particular, $J(E_Q) \subseteq \nabla E_Q = \mathrm{Tot}(E_Q)$ *and* $\nabla E_Q \subseteq \Delta E_Q$.

Proof. (1). *By Kasch [4].* (2). *Since* $J[Q, N] \subseteq \mathrm{Tot}[Q, N]$, *so by* (1) $J[Q, N] \subseteq \Delta[Q, N]$. (3). *Let* $\alpha \in \nabla[Q, N]$ *and suppose that* $\alpha \notin \Delta[Q, N]$ *then* $Ker(\alpha)$ *is not large in* Q, *so there exists an injective module* $0 \neq A \subseteq Q$ *such that* $A \cap Ker(\alpha) = 0$. *Since* A *is injective there exists* $\beta : N \to A$ *such that* $\beta \alpha|_A = I_A$ *so* $\beta = \beta\alpha\beta$. *Thus* $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$, $Im(\alpha\beta) \subseteq^{\oplus} N$ *and* $Im(\alpha\beta) \subseteq Im(\alpha)$ $<< N$, *so* $Im(\alpha\beta) = 0$ *and* $\alpha\beta = 0$, *a contradiction. Thus* $\alpha \in \Delta[Q, N]$.

Y. Zhou gave an example of a locally injective module which does not have a semipotent endomorphism ring [9, Example 4.2]. The following Theorem gave us a necessary and sufficient conditions for endomorphism ring of a locally injective module to be semipotent ring.

Theorem 3.2. *Let* Q_R *be a locally injective module. For any module* $N \in mod - R$ *the following are equivalent:*

(1) $[Q, N]$ *is a semipotent.*

(2) $\text{Tot}[Q,N] = J[Q,N] = \Delta[Q,N]$.

(3) *For any* $\alpha \in [Q,N] \setminus J[Q,N]$ *there exists* $\beta \in [N,Q]$ *with* $0 \neq Ker(\beta\alpha) \subseteq^{\oplus} Q$.

In particular, E_Q *is a semipotent ring if and only if, for any* $\alpha \in E_Q \setminus J(E_Q)$ *there exists* $0 \neq \beta \in E_Q$ *such that* $Ker(\beta\alpha) \subseteq^{\oplus} Q$.

Proof. $(1) \Rightarrow (3)$. $\alpha \in [Q,N] \setminus J[Q,N]$ *then there exists* $\beta \in [N,Q]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_Q$, *so* $0 \neq Ker(\beta\alpha) \subseteq^{\oplus} Q$.

$(3) \Rightarrow (2)$. *Since* Q *is a locally injective then by Lemma 3.1* $J[Q,N] \subseteq \Delta[Q,N]$. *Let* $\alpha \in \Delta[Q,N]$ *and suppose that* $\alpha \notin J[Q,N]$ *then there exists* $\beta \in [N,Q]$ *such that* $0 \neq Ker(\beta\alpha) \subseteq^{\oplus} Q$ *and* $Ker(\alpha) \subseteq Ker(\beta\alpha)$. *Since* $Ker(\alpha) \leq_e Q$ *then* $Ker(\beta\alpha) \leq_e Q$ *and* $Ker(\beta\alpha) \cap Im(\beta\alpha) = 0$ *so* $Im(\beta\alpha) = 0$ *and* $\beta\alpha = 0$, *a contradiction. Thus,* $\alpha \in J[Q,N]$.

$(2) \Rightarrow (1)$. *Since* $J[Q,N] = \Delta[Q,N] = \text{Tot}[Q,N]$ *then by [9, Theorem 2.2]* $[Q,N]$ *is semipotent.*

Lemma 3.3. *Let* M_R, N_R *be modules. The following are equivalent:*

(1) *If* $\alpha \in [M,N] \setminus \Delta[M,N]$ *there exists* $\beta \in [N,M]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. (2) *If* $\alpha \in [M,N] \setminus \Delta[M,N]$ *there exists* $\beta \in [N,M]$ *such that* $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$. (3) *If* $\alpha \in [M,N] \setminus \Delta[M,N]$ *there exists* $\gamma \in [N,M]$ *such that* $\gamma\alpha\gamma = \gamma \notin \Delta[N,M]$. *Proof.* $(1) \Rightarrow (3)$. *Let* $\alpha \in [M,N] \setminus \Delta[M,N]$, *then* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ *for some* $\beta \in [N,M]$. *Let* $\gamma = \beta\alpha\beta$ *then* $\gamma \in [N,M]$ *and* $\gamma\alpha\gamma = \gamma \notin \Delta[N,M]$ *because* $\beta\alpha \notin \Delta E_M$. *Suppose* (3) *holds، let* $\alpha \in [M,N] \setminus \Delta[M,N]$ *then* $\gamma = \gamma\alpha\gamma$ *for some* $\gamma \in [N,M] \setminus \Delta[N,M]$ *so* $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$, *gives* (1). *Similarly, the equivalence* $(2) \Leftrightarrow (3)$ *holds.*

We call that $[M, N]$ *is* $\Delta-$*semipotent if the conditions in lemma 3.3 are satisfied. In particular,* E_M *is a* $\Delta-$*semipotent ring if and only if,* $[M, M]$ *is a* $\Delta-$ *semipotent.*

Proposition 3.4. *Let* Q_R *be a locally injective module, then the following hold:*

(1) E_Q *is a* $\Delta-$ *semipotent ring, where* $\Delta = \Delta E_Q \subseteq E_Q$.

(2) $[Q, N]$ *is an* $\Delta-$ *semipotent for any module* $N \in mod-R$.

Proof. (1). *Let* $\alpha \in E_Q$, $\alpha \notin \Delta E_Q$, *then* $Ker(\alpha)$ *is not large in* Q. *So, there exists an injective submodule* $0 \neq K \subseteq Q$ *with* $K \cap Ker(\alpha) = 0$. *Since* K *is an injective module there exist* $\beta : Q \to K$ *such that* $\beta\alpha\,|_K = I_K$. *Hence, for any* $x \in Q$, $\beta(x) \in K$ *so,* $\beta\alpha\beta(x) = \beta(x)$ *for all* $x \in Q$ *therefore,* $\beta\alpha\beta = \beta$. *Thus,* $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_Q$. *This shows that* E_Q *is a* $\Delta-$ *semipotent ring.*

(2). *Let* $\alpha \in [Q, N] \setminus \Delta[Q, N]$ *then* $Ker(\alpha)$ *is not large in* Q. *Since* Q *is a locally injective, there exists an injective submodule* $0 \neq A \subseteq Q$ *such that* $A \cap Ker(\alpha) = 0$. *Since* A *is injective there exists* $\beta : A \to N$ *such that* $\beta\alpha\,|_A = I_A$. *Hence, for any* $y \in N$, $\beta(y) \in A$ *then* $\beta\alpha\beta(y) = \beta(y)$ *for all* $y \in N$, *thus* $\beta\alpha\beta = \beta$. *Therefore,* $\beta \in [N, Q]$ *such that* $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_N$. *This shows that* $[Q, N]$ *is* $\Delta-$*semipotent.*

Corollary 3.5. *The following hold:*

(a) *Let* P_R *be a locally projective module then:*

(1) E_P *is an* $I-$ *(or,* $\nabla-$*) semipotent ring.*

(2) $\mathrm{Tot}[M, P] = \nabla[M, P] = I[M, P]$ *for any module* $M \in mod-R$.

(3) $[M, P]$ *is* $I-$ *(or,* $\nabla-$*) semipotent for any module* $M \in mod-R$.

(b) *Let* Q_R *be a locally injective module then:*

(1) E_Q *is a* $\Delta-$ *semipotent ring.*

(2) $\mathrm{Tot}[Q,N]=\Delta[Q,N]$ *for any module* $N\in mod-R$.

(3) $[Q,N]$ *is* $\Delta-$ *semipotent for any module* $N\in mod-R$.

Proof. (a). (1) *by Proposition 2.12.* (2) *By lemma 2.6.* (3) *by proposition 2.12.*

(b). (1) *by proposition 3.4.* (2) *by lemma 3.1.* (3) *by proposition 3.4.*

Theorem 3.6. *Let* P_R *be a locally projective module and* Q_R *be a locally injective module, then the following hold:*

(1) $\mathrm{Tot}[Q,P]=\Delta[Q,P]=\nabla[Q,P]=I[Q,P]$.

(2) $\mathrm{Tot}[Q,P]=J[Q,P]$.

(3) $[Q,P]$ *is semipotent.*

In particular, if M_R *is a locally projective and locally injective module then* $\mathrm{Tot}(E_M)=J(E_M)$ *and* E_M *is a semipotent ring.*

Proof. (1). *Since* P *is a locally projective module then by lemma 2.6 we have* $\mathrm{Tot}[Q,P]=\nabla[Q,P]=I[Q,P]$. *On the other hand, since* Q *is a locally injective module then by lemma 3.1, we have* $\mathrm{Tot}[Q,P]=\Delta[Q,P]$. *Thus,* $\mathrm{Tot}[Q,P]=\Delta[Q,P]=\nabla[Q,P]=I[Q,P]$.

(2). *It is clear that* $J[Q,P]\subseteq\mathrm{Tot}[Q,P]$. *Let* $\alpha\in\mathrm{Tot}[Q,P]$ *then by (1)* $\alpha\in\Delta[Q,P]=\nabla[Q,P]$ *so,* $Im(\alpha)<<P$ *and* $Ker(\alpha)\leq_e Q$. *Thus, for any* $\beta\in[P,Q]$; $\beta\alpha\in E_Q$, $Im(\beta\alpha)<<Q$, *hence* $Im(\beta\alpha)=\beta(Im(\alpha))$ *and* $Ker(\beta\alpha)\leq_e Q$, *hence* $Ker(\alpha)\subseteq Ker(\beta\alpha)$. *Since* $Q=Im(\beta\alpha)+Im(1_Q-\beta\alpha)$ *follows that* $Q=Im(1_Q-\beta\alpha)$ *and* $Ker(1_Q-\beta\alpha)=0$, *hence* $Ker(\beta\alpha)\cap Ker(1_Q-\beta\alpha)=0$, *therefore* $1_Q-\beta\alpha\in U(E_Q)$. *Thus,* $\alpha\in J[Q,P]$. (3). *By (2) and [9, Theorem 2.2].*

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