Locally Projective and Locally Injective Modules II

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ABSTRACT

The object of this paper is to study the locally projective and locally injective modules. Specifically, this paper is a continuation of study of locally projective and locally injective modules, where a new description of locally projective and locally injective modules is obtained. The main obtained results include:

(a) ***Let*** P_R ***be a module. The following conditions are equivalent:***

(1) P ***is a locally projective module.***

(2) $\mathrm{Tot}[M,P]=\nabla[M,P]$ ***for all*** $M\in mod-R$.

(3) $\mathrm{Tot}[P,M]=\nabla[P,M]$ ***for all*** $M\in mod-R$.

(4) $[P,M]$ ***is a*** $\nabla-$***semipotent for all*** $M\in mod-R$.

(5) $[M,P]$ ***is a*** $\nabla-$***semipotent for all*** $M\in mod-R$.

(b) ***Let*** Q_R ***be a module. The following conditions are equivalent:***

(1) Q ***is a locally injective module.***

(2) $\mathrm{Tot}[Q,N]=\Delta[Q,N]$ ***for all*** $N\in mod-R$.

(3) $\mathrm{Tot}[N,Q]=\Delta[N,Q]$ ***for all*** $N\in mod-R$.

(4) $[Q,N]$ ***is a*** $\Delta-$***semipotent for all*** $N\in mod-R$.

(5) $[N,Q]$ ***is a*** $\Delta-$***semipotent for all*** $N\in mod-R$.

(c) ***The following conditions are equivalent for any ring*** R ***:***

(1) ***Every module*** $M\in mod-R$ ***with*** E_M ***is a*** $\Delta-$***semipotent ring, is injective.***

(2) $R-$***is a semi-simple Artinian ring.***

(3)***Every module*** $M\in mod-R$ ***with*** E_M ***is a*** $\nabla-$***semipotent ring, is projective.***

Key Words*:* *Semipotent Rings, Locally projective and locally injective modules, The total, Jacobson radical, (co) singular ideal, Endomorphisms rings,* $hom_R(M,N)$.

MSC 2010. $16E50, 16E60, 16D70$.

المودولات الإسقاطية المحلية والأفقية المحلية (II)

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الملخص

الهدف من هذا البحث هو دراسة المودولات الإسقاطية المحلية والأفقية المحلية. بشكل خاص، تعـد هذه الورقة متابعة لدراسة المودولات الإسقاطية والأفقية المحلية للحصول علـى وصـف جديـد لهـذه المودولات. النتائج الرئيسية التي تم التوصل إليها من خلال هذه الدراسة هي:

(أ) ليكن P_R مودولاً فوق الحلقة R . الشروط التالية متكافئة:

1. **المودول P يكون مودولاً إسقاطياً محلياً.**
2. **لأجل أي مودول $M \in mod - R$ فإن $\text{Tot}[M,P] = \nabla[M,P]$.**
3. **لأجل أي مودول $M \in mod - R$ فإن $\text{Tot}[P,M] = \nabla[P,M]$.**
4. **لأجل أي مودول $M \in mod - R$ المودول الثنائي $[P,M]$ يكون $\nabla -$ شبه جامد.**
5. **لأجل أي مودول $M \in mod - R$ المودول الثنائي $[M,P]$ يكون $\nabla -$ شبه جامد.**

(ب) ليكن Q_R مودولاً فوق الحلقة R . الشروط التالية متكافئة:

1. **المودول Q يكون مودولاً أفقياً محلياً.**
2. **لأجل أي مودول $N \in mod - R$ فإن $\text{Tot}[Q,N] = \Delta[Q,N]$.**
3. **لأجل أي مودول $N \in mod - R$ فإن $\text{Tot}[N,Q] = \Delta[N,Q]$.**
4. **لأجل أي مودول $N \in mod - R$ المودول الثنائي $[Q,N]$ يكون $\Delta -$ شبه جامد.**
5. **لأجل أي مودول $N \in mod - R$ المودول الثنائي $[N,Q]$ يكون $\Delta -$ شبه جامد.**

(ج) لأجل أي حلقة R الشروط التالية متكافئة:

1. **أي مودول $M \in mod - R$ من أجله الحلقة E_M هي $\Delta -$ شبه جامدة، يكون مودولاً محلياً.**
2. **الحلقة R تكون نصف بسيطة أرتينية.**
3. **أي مودول $M \in mod - R$ من أجله الحلقة E_M هي $\nabla -$ شبه جامدة، يكون مودولاً إسقاطياً.**

الكلمات المفتاحية:الحلقات شبه الجامدة، المودولات الإسقاطية المحليـة، المـودولات الأفقية المحلية، التوتال، أساس جاكبسون، المثالي المنفرد، حلقـة التشاكلات لمودول، $hom_R(M,N)$.

رقم التصنيف العالمي: $16E50, 16D40$.

1. Introduction.

In this paper the ring R *is associative with identity unless otherwise indicated. All modules over a ring* R *are unitary right modules. A submodule* N *of a module* M *is said to be small in* M *if* $N+K \neq M$ *for any proper submodule* K *of* M *, [3]. A submodule* N *of a module* M *is said to be large (essential) in* M *if* $N \cap K \neq 0$ *for any nonzero submodule* K *of* M *, [3]. If* M *is an* $R-$*module, the radical of* M *denoted by* $J(M)$ *is defined to be the intersection of all maximal submodules of M. Also,* $J(M)$ *coincides with the sum of all small submodules of* M *,[5]. It my happen that* M *has no maximal submodules in which case* $J(M)=M$ *, [5]. Thus, for a ring* R*,* $J(R)$ *is the Jacobson radical of* R*. Also, we write* $U(R)$ *the group of units of a ring* R*. For a submodule* N *of a module M, we use* $N \subseteq^{\oplus} M$ *to mean that* N *is a direct summand of M, and we write* $N \leq_e M$ *and* $N << M$ *to indicate that* N *is a large, respectively small, submodule of* M *.*

If M_R *is a module, we use the notation* $E_M = End_R(M)$ *and we write*

$$\Delta E_M = \{a : a \in E_M ; Ker(a) \leq_e M\}, \nabla E_M = \{a : a \in E_M ; Im(a) << M\}$$

And $I(E_M) = \{a : a \in E_M ; Im(a) \subseteq J(M)\}$. *It is well known that* $\Delta E_M, \nabla E_M$ *and* $I(E_M)$ *are ideals in* E_M *[3]. If* M_R *and* N_R *are modules, we use* $[M,N]=$ $hom_R(M,N)$. *Thus,* $[M,N]$ *is an* $(E_M, E_N)-$*bimodule. Our main concern is about the substructures of* $hom_R(M,N)$ *and the semipotent of* $hom_R(M,N)$ *(see [6]).*

Following [6], let M_R, N_R *be modules. We putI* $[M,N]=hom_R(M,N)$. *Thus,* $[M,N]$ *is an* (E_M, E_N)*-bimodule. The four substructures of* $hom_R(M,N)$ *are given as follows (see [6]).*

- *The Jacobson radical*

$$J[M,N] = \{a : a \in [M,N];\ b\,a \in J(E_M) \ \text{ for all } \ b \in [N,M]\}$$
$$J[M,N] = \{a : a \in [M,N];\ a\,b \in J(E_N) \ \text{ for all } \ b \in [N,M]\}$$
$$J[M,N] = \{a : a \in [M,N];\ 1_N - a\,b \in U(E_N) \ \text{ for all } \ b \in [N,M]\}$$

$J[M,N]=\{a : a\in[M,N];\ 1_M - b\,a\in U(E_M)\ \ for\ all\ \ b\in[N,M]\}$.

Thus, $J[M,M]=J(E_M)$. *In particular,* $J[R,R]=J(R)$.

- *The singular ideal*

$\Delta[M,N]=\{a : a\in[M,N];\ Ker(a)\le_e M\}$.

In particular, $\Delta[M,M]=\Delta E_M$.

- *The co-singular ideal*

$\nabla[M,N]=\{a : a\in[M,N];\ Im(a) << N\}$.

In particular, $\nabla[M,M]=\nabla E_M$.

- *The total*

$\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ a\ [N,M]\ contains\ no\ nonzero\ idempotents\}$

$\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ [N,M]a\ \ contains\ no\ nonzero\ idempotents\}$

Thus,

$\mathrm{Tot}[M,M]=\mathrm{Tot}(E_M)=\{a : a\in E_M;\ a\,E_M\ contains\ no\ nonzero\ idempotents\}$

$=\{a : a\in E_M;\ E_M\,a\ \ contains\ no\ nonzero\ idempotents\}$.

2. Locally Projective Modules.

A module P_R *is called locally projective [4] if, for every submodule* $B\subseteq P$, *which is not small in* P *there exists a projective direct summand* $0\ne W\subseteq^{\oplus} P$ *with* $W\subseteq B$.

We present in the following, a number of results from [2] which will be used throughout the paper.

Lemma 2.1. *[2, Lemma 4.1]. Let* M_R, N_R *be modules then:*

(1) $\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ b\,a\in\mathrm{Tot}(E_M)\ \ for\ all\ \ b\in[N,M]\}$.

(2) $\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ a\ b\in\mathrm{Tot}(E_N)\ \ for\ all\ \ b\in[N,M]\}$.

Lemma 2.2. *[2, Lemma 5.6]. Let* M_R, N_R *be modules. The following are equivalent:*

(1) *If* $a\in[M,N]\setminus\nabla[M,N]$ *there exists* $b\in[N,M]$ *such that* $0\ne ba=(ba)^2\in E_M$.

(2) *If* $a \in [M,N] \setminus \nabla[M,N]$ *there exists* $b \in [N,M]$ *such that* $0 \neq ab = (ab)^2 \in E_N$.

(3) *If* $a \in [M,N] \setminus \nabla[M,N]$ *there exists* $g \in [N,M]$ *such that* $g\,a\,g = g \notin \nabla[N,M]$.

We call that $[M,N]$ *is* $\nabla-$*semipotent [2], if the conditions in lemma 2.2 are satisfied.*

Lemma 2.3. *[2, Theorem 5.7]. Let* M_R, N_R *be modules.* $[M,N]$ *is* $\nabla-$ *semipotent if and only if,* $\mathrm{Tot}[M,N] = \nabla[M,N]$. *In particular,* E_M *is a* $\nabla-$ *semipotent ring if and only if,* $\mathrm{Tot}(E_M) = \nabla E_M$.

Proposition 2.4. *Let* M_R *be module. The following conditions are equivalent:*

(1) $[M,N]$ *is* $\nabla-$*semipotent for any module* $N \in mod - R$.

(2) E_M *is a* $\nabla-$*semipotent ring and for any module* $N \in mod - R$, $\nabla[M,N] = \{a : a \in [M,N];\, ba \in \nabla E_M \text{ for any } b \in [N,M]\}$.

Proof. $(1) \Rightarrow (2)$. *Since for any module* $N \in mod - R$, $[M,N]$ *is* $\nabla-$*semipotent then by lemma 2.3,* $\mathrm{Tot}[M,N] = \nabla[M,N]$, *by [2, Proposition 5.5] we have* $\nabla[M,N] = \{a : a \in [M,N];\, b\,a \in \nabla E_M$ *for all* $b \in [N,M]\}$. *On the other hand, we have* $\mathrm{Tot}\,(E_M) = \mathrm{Tot}[M,M] = \nabla[M,M] = \nabla E_M$, *so by lemma 2.3* E_M *is a* $\nabla-$*semi-potent ring.*

$(2) \Rightarrow (1)$. *Let* N_R *be a module and* $a \in \nabla[M,N]$. *Suppose* $a \notin \mathrm{Tot}[M,N]$ *then* $0 \neq (b\,a)^2 = b\,a \in E_M$ *for some* $b \in [N,M]$, *so* $Im(b\,a) \oplus Ker(b\,a) = M$. *On the other hand, since* $a \in \nabla[M,N]$ *then* $Im(a) << N$ *and* $Im(b\,a) << M$, *thus* $Ker(b\,a) = M$, *so* $b\,a = 0$ *a contradiction. Hence* $\nabla[M,N] \subseteq \mathrm{Tot}[M,N]$.

Let $a \in \mathrm{Tot}[M,N]$ *then by Lemma 2.1,* $b\,a \in \mathrm{Tot}(E_M)$ *for all* $b \in [N,M]$. *By (2) since* E_M *is a* $\nabla-$ *semipotent ring then by lemma*

2.3, $\mathrm{Tot}(E_M)=\nabla E_M$*, so* $ba\in\nabla E_M$ *for all* $b\in[N,M]$ *thus, by assumption* $a\in\nabla[M,N]$*. So* $\mathrm{Tot}[M,N]\subseteq\nabla[M,N]$*. Thus we have* $\mathrm{Tot}[M,N]=\nabla[M,N]$ *by lemma 2.3* $[M,N]$ *is* $\nabla-$*semipotent for any module* $N\in mod-R$.

Proposition 2.5. *Let* N_R *be module. The following conditions are equivalent:*

(1) $[M,N]$ *is* $\nabla-$*semipotent for any module* $M\in mod-R$.

(2) E_N *is a* $\nabla-$*semipotent ring and for any module* $M\in mod-R$,

$\nabla[M,N]=\{a : a\in[M,N]; ab\in\nabla E_N$ *for all* $b\in[N,M]\}$.

Proof. $(1)\Rightarrow(2)$. *Since for any module* $M\in mod-R$, $[M,N]$ *is* $\nabla-$*semipotent then by lemma 2.3,* $\mathrm{Tot}[M,N]=\nabla[M,N]$*, by [2, Proposition 5.5] we have* $\nabla[M,N]=\{a : a\in[M,N]; ab\in\nabla E_N$ *for all* $b\in[N,M]\}$*. On the other hand, we have* $\mathrm{Tot}(E_N)=\mathrm{Tot}[N,N]=\nabla[N,N]=\nabla E_N$*, so by lemma 2.3* E_N *is a* $\nabla-$*semi-potent ring.*

$(2)\Rightarrow(1)$*. Let* M_R *be a module and* $a\in\nabla[M,N]$*. Suppose* $a\notin\mathrm{Tot}[M,N]$ *then* $0\neq(ab)^2=ab\in E_N$ *for some* $b\in[N,M]$*, so* $Im(ab)\oplus Ker(ab)=N$*. On the other hand, since* $a\in\nabla[M,N]$ *then* $Im(a)<<N$*, since* $Im(ab)\subseteq Im(a)$ *then* $Im(ab)<<N$*, thus* $Ker(ab)=N$*, so* $ab=0$ *a contradiction. Hence* $\nabla[M,N]\subseteq\mathrm{Tot}[M,N]$.

Let $a\in\mathrm{Tot}[M,N]$ *then by Lemma 2.1,* $ab\in\mathrm{Tot}(E_N)$ *for all* $b\in[N,M]$*. By (2) since* E_N *is a* $\nabla-$ *semipotent ring then by lemma 2.3,* $\mathrm{Tot}(E_N)=\nabla E_N$*, so* $ab\in\nabla E_N$ *for all* $b\in[N,M]$ *thus, by assumption* $a\in\nabla[M,N]$*. So* $\mathrm{Tot}[M,N]\subseteq\nabla[M,N]$*. Thus we have* $\mathrm{Tot}[M,N]=\nabla[M,N]$ *by lemma 2.3* $[M,N]$ *is* $\nabla-$*semipotent for any module* $M\in mod-R$.

Corollary 2.6. *The following conditions are equivalent:*

(1) *For any module* $M \in mod-R$, $[M,N]$ *is* $\nabla-$*semipotent for all* $N \in mod-R$.

(2) *For any module* $N \in mod-R$, $[M,N]$ *is* $\nabla-$*semipotent for all* $M \in mod-R$.

Proof. $(1) \Rightarrow (2)$. *Let* N_R *be module. Suppose* $M \in mod-R$, *then by (1)* $[M,N]$ *is* $\nabla-$*semipotent, and by lemma 2.3 we have* $\mathrm{Tot}[M,N] = \nabla[M,N]$. *So by [2, proposition 5.5 (b)] we have*

$\nabla[M,N] = \{a : a \in [M,N]; ab \in \nabla E_N$ *for all* $b \in [N,M]\}$

And E_N *is a* $\nabla-$*semipotent ring.*

$(2) \Rightarrow (1)$. *Let* M_R *be module. Suppose* $N \in mod-R$, *then by (2)* $[M,N]$ *is* $\nabla-$*semipotent, and by lemma 2.3 we have* $\mathrm{Tot}[M,N] = \nabla[M,N]$. *So by [2, proposition 5.5 (b)] we have*

$\nabla[M,N] = \{a : a \in [M,N]; ba \in \nabla E_M$ *for any* $b \in [N,M]\}$

And E_M *is a* $\nabla-$*semipotentring.*

Theorem 2.7. *Let* P_R *be a module. The following conditions are equivalent:*

(1) P *is a locally projective module.*

(2) $\mathrm{Tot}[M,P] = \nabla[M,P]$ *for all* $M \in mod-R$.

(3) $\mathrm{Tot}[P,M] = \nabla[P,M]$ *for all* $M \in mod-R$.

(4) $[P,M]$ *is a* $\nabla-$*semipotent for all* $M \in mod-R$.

(5) $[M,P]$ *is a* $\nabla-$*semipotent for all* $M \in mod-R$.

Proof. $(1) \Leftrightarrow (2)$ *by Kasch [4].* $(2) \Leftrightarrow (3)$ *by Corollary 2.6.* $(3) \Leftrightarrow (4)$ *and* $(2) \Leftrightarrow (5)$ *by lemma 2.3.*

Proposition 2.8. *Let* P_R *be a locally projective module. For any module* $M \in mod-R$ *the following conditions are equivalent:*

(1) $[M,P]$ *is semipotent.*

(2) $\operatorname{Tot}[M,P]=\nabla[M,P]=J[M,P]$.

Proof. $(1)\Rightarrow(2)$. *Suppose that* $[M,P]$ *is semipotent then by [6, Theorem 2.2]*

$\operatorname{Tot}[M,P]=J[M,P]$. *Since P is locally projective then by Kasch [4]*

$\operatorname{Tot}[M,P]=\nabla[M,P]$. *Thus,* $\operatorname{Tot}[M,P]=\nabla[M,P]=J[M,P]$.

$(2)\Rightarrow(1)$. *Since* $\operatorname{Tot}[M,P]=J[M,P]$ *then by [6, Theorem 2.2]* $[M,P]$ *is semipotent.*

3. Locally injective Modules.

A module Q_R *is called locally projective [4] if, for every submodule* $A\subseteq Q$, *which is not large in* Q *there exists an injective summand* $0\neq B\subseteq Q$ *with* $A\cap B=0$.

Lemma 3.1. *[2, Lemma 5.2]. Let* M_R, N_R *be modules. The following are equivalent:*

(1) *If* $a\in[M,N]\setminus\Delta[M,N]$ *there exists* $b\in[N,M]$ *such that* $0\neq ba=(ba)^2\in E_M$. (2) *If* $a\in[M,N]\setminus\Delta[M,N]$ *there exists* $b\in[N,M]$ *such that* $0\neq ab=(ab)^2\in E_N$. (3) *If* $a\in[M,N]\setminus\Delta[M,N]$ *there exists* $g\in[N,M]$ *such that* $g\,a\,g=g\notin\Delta[N,M]$.

We call that $[M,N]$ *is* $\Delta-$*semipotent [2], if the conditions in lemma 3.1 are satisfied.*

Lemma 3.2. *[2, Theorem 5.3]. Let* M_R, N_R *be modules.* $[M,N]$ *is* $\Delta-$*semipotent if and only if,* $\operatorname{Tot}[M,N]=\Delta[M,N]$. *In particular,* E_M *is a* $\Delta-$*semipotent ring if and only if,* $\operatorname{Tot}(E_M)=\Delta E_M$.

Proposition 3.3. *Let* M_R *be module. The following conditions are equivalent:*

(1) $[M,N]$ *is* $\Delta-$*semipotent for any module* $N\in mod-R$.

(2) E_M *is a* $\Delta-$*semipotent ring and for any module* $N\in mod-R$,

$\Delta[M, N] = \{a : a \in [M, N];\ ba \in \Delta E_M$ *for any* $b \in [N, M]\}$.

Proof. $(1) \Rightarrow (2)$. *Since for any module* $N \in mod - R$, $[M, N]$ *is* $\Delta -$*semipotent then by lemma 3.2,* $\mathrm{Tot}[M, N] = \Delta[M, N]$, *by [2, Proposition 5.1(b)] we have* $\Delta[M, N] = \{a : a \in [M, N];\ b\,a \in \Delta E_M$ *for all* $b \in [N, M]\}$. *On the other hand, we have* $\mathrm{Tot}\,(E_M) = \mathrm{Tot}[M, M] = \Delta[M, M] = \Delta\, E_M$, *so by lemma 3.2* E_M *is a* $\Delta -$*semipotent ring.*

$(2) \Rightarrow (1)$. *Let* N_R *be a module and* $a \in \Delta[M, N]$. *Suppose* $a \notin \mathrm{Tot}[M, N]$ *then* $0 \neq (b\,a)^2 = b\,a \in E_M$ *for some* $b \in [N, M]$, *so* $Im\,(b\,a) \oplus Ker\,(b\,a) = M$. *On the other hand, since* $a \in \Delta[M, N]$ *then* $Ker\,(a) \leq_e M$, *so* $Ker\,(ba) \leq_e N$ *hence* $Ker\,(a) \subseteq Ker\,(b\,a)$. *Since* $Im(b\,a) \cap Ker\,(b\,a) = 0$ *follows* $Im(b\,a) = 0$, *so* $b\,a = 0$ *a contradiction. Hence* $\Delta[M, N] \subseteq \mathrm{Tot}[M, N]$.

Let $a \in \mathrm{Tot}[M, N]$ *then by Lemma 2.1,* $b\,a \in \mathrm{Tot}\,(E_M)$ *for all* $b \in [N, M]$. *By (2) since* E_M *is a* $\Delta -$*semipotent ring then by lemma 3.2,* $\mathrm{Tot}(E_M) = \Delta\, E_M$, *so* $b\,a \in \Delta\, E_M$ *for all* $b \in [N, M]$ *thus, by assumption* $a \in \Delta[M, N]$. *So* $\mathrm{Tot}[M, N] \subseteq \Delta[M, N]$. *Thus we have* $\mathrm{Tot}[M, N] = \Delta[M, N]$ *by lemma 3.2* $[M, N]$ *is* $\Delta -$*semipotent for any module* $N \in mod - R$.

Proposition 3.4. *Let* N_R *be module. The following conditions are equivalent:*

(1) $[M, N]$ *is* $\Delta -$*semipotent for any module* $M \in mod - R$.

(2) E_N *is a* $\Delta -$*semipotent ring and for any module* $M \in mod - R$,

$\Delta[M, N] = \{a : a \in [M,N];\ a\,b \in \Delta E_N$ *for all* $b \in [N, M]\}$.

Proof. $(1) \Rightarrow (2)$. *Since for any module* $M \in mod - R$, $[M, N]$ *is* $\Delta -$*semipotent then by lemma 3.2,* $\mathrm{Tot}[M, N] = \Delta[M, N]$, *by [2, Proposition 5.1(b)] we have* $\Delta[M, N] = \{a : a \in [M, N];\ a\,b \in \Delta\, E_N$ *for all* $b \in [N, M]\}$. *On the other hand, we have*

$\text{Tot}(E_N) = \text{Tot}[N,N] = \Delta[N,N] = \Delta E_N$, *so by lemma 3.2,* E_N *is a* $\Delta-$*semi-potent ring.*

$(2) \Rightarrow (1)$. *Let* M_R *be a module and* $a \in \Delta[M,N]$. *Suppose* $a \notin \text{Tot}[M,N]$ *then* $0 \neq (ab)^2 = ab \in E_N$ *for some* $b \in [N,M]$, *so* $Im(ab) \oplus Ker(ab) = N$. *On the other hand, since* $a \in \Delta[M,N]$ *then* $ab \in \Delta E_N$, *therefore* $Ker(ab) \leq_e N$. *Hence* $Im(ab) \cap Ker(ab) = 0$ *follows* $ab = 0$ *a contradiction. Thus,* $a \in \text{Tot}[M,N]$ *and* $\Delta[M,N] \subseteq \text{Tot}[M,N]$.

Let $a \in \text{Tot}[M,N]$ *then by Lemma 2.1,* $ab \in \text{Tot}(E_N)$ *for all* $b \in [N,M]$. *Since by (2)* E_N *is a* $\Delta-$ *semipotent ring then by lemma 3.2,* $\text{Tot}(E_N) = \Delta E_N$ *, so* $ab \in \Delta E_N$ *for all* $b \in [N,M]$ *thus, by assumption* $a \in \Delta[M,N]$. *So* $\text{Tot}[M,N] \subseteq \Delta[M,N]$. *Thus we have* $\text{Tot}[M,N] = \Delta[M,N]$ *by lemma 3.2* $[M,N]$ *is* $\Delta-$*semipotent for any module* $M \in mod-R$.

Corollary 3.5. *The following conditions are equivalent:*

(1) *For any module* $M \in mod-R$, $[M,N]$ *is* $\Delta-$*semipotent for all* $N \in mod-R$.

(2) *For any module* $N \in mod-R$, $[M,N]$ *is* $\Delta-$*semipotent for all* $M \in mod-R$.

Proof. $(1) \Rightarrow (2)$. *Let* N_R *be module. Suppose* $M \in mod-R$, *then by (1)* $[M,N]$ *is* $\Delta-$*semipotent, and by lemma 3.2 we have* $\text{Tot}[M,N] = \Delta[M,N]$. *So by [2, proposition 5.1 (b)] we have*

$$\Delta[M,N] = \{a : a \in [M,N];\ ab \in \Delta E_N \text{ for all } b \in [N,M]\}$$

And E_N *is a* $\Delta-$*semipotent ring.*

$(2) \Rightarrow (1)$. *Let* M_R *be module. Suppose* $N \in mod-R$, *then by (2)* $[M,N]$ *is* $\Delta-$*semipotent, and by lemma 3.2 we have* $\text{Tot}[M,N] = \Delta[M,N]$. *So by [2, proposition 5.1 (b)] we have*

$$\Delta[M,N] = \{a : a \in [M,N];\ ba \in \Delta E_M \text{ for any } b \in [N,M]\}$$

And E_M *is a* $\Delta -$ *semipotentring.*

Theorem 3.6. *Let* Q_R *be a module. The following conditions are equivalent:*

(1) Q *is a locally injective module.*

(2) $\mathrm{Tot}[Q,N]=\Delta[Q,N]$ *for all* $N \in mod-R$.

(3) $\mathrm{Tot}[N,Q]=\Delta[N,Q]$ *for all* $N \in mod-R$.

(4) $[Q,N]$ *is a* $\Delta -$ *semipotent for all* $N \in mod-R$.

(5) $[N,Q]$ *is a* $\Delta -$ *semipotent for all* $N \in mod-R$.

Proof. $(1) \Leftrightarrow (2)$ *by Kasch [4].* $(2) \Leftrightarrow (3)$ *by Corollary 3.5.* $(3) \Leftrightarrow (4)$ *and* $(2) \Leftrightarrow (5)$ *by lemma 3.2.*

Proposition 3.7. *Let* Q_R *be a locally injective module. For any module* $N \in mod-R$ *the following conditions are equivalent:*

(1) $[Q,N]$ *is semipotent.*

(2) $\mathrm{Tot}[Q,N]=\Delta[Q,N]=J[Q,N]$.

Proof. $(1) \Rightarrow (2)$. *Suppose that* $[Q,N]$ *is semipotent then by [6, Theorem 2.2]* $\mathrm{Tot}[Q,N]=J[Q,N]$. *Since* Q *is locally injective then by Kasch [4]* Q. *Thus,* $\mathrm{Tot}[M,P]=\Delta[M,P]$. $(2) \Rightarrow (1)$. *Since* $\mathrm{Tot}[M,P]=J[M,P]$ *then by [6, Theorem 2.2]* $[Q,N]$ *is semipotent. Let*

$$\Phi(R)=\{M \in mod-R : \mathrm{Tot}[M,N]=J[M,N], \text{for any } N \in mod-R\}$$

$$\Gamma(R)=\{N \in mod-R : \mathrm{Tot}[M,N]=J[M,N], \text{for any } M \in mod-R\}$$

$$\Delta\Phi(R)=\{M \in mod-R : \mathrm{Tot}[M,N]=\Delta[M,N], \text{for any } N \in mod-R\}$$

$$\nabla\Gamma(R)=\{N \in mod-R : \mathrm{Tot}[M,N]=\nabla[M,N], \text{for any } M \in mod-R\}$$

Theorem3.8. *The following conditions are equivalent for any ring* R *:*

(1) *Every module* $M \in mod-R$ *with* E_M *is a* $\Delta -$ *semipotent ring, is injective.*

(2) $\Phi(R) = \Delta\Phi(R)$.

(3) *Every module* $M \in mod - R$ *with* E_M *is semipotent ring, is injective.*

(4) $R-$ *is a semi-simple Artinian ring.*

(5) *Every module* $M \in mod - R$ *with* E_M *is semipotent ring, is projective.*

(6) $\Gamma(R) = \nabla\Gamma(R)$.

(7) *Every module* $M \in mod - R$ *with* E_M *is a* $\nabla-$*semipotent ring,is projective.*

Proof. By [6, Corollary 4.7] and propositions 3.7 and 2.8.

Corollary 3.9. *The following conditions are equivalent for any ring* R *:*

(1) R *is a semipotent ring and* $J(R)$ *is left* $T-$*nilpotent.*

(2) E_P *is a semipotent ring for every projective module* $P \in mod - R$.

(3) E_P *is a* $\nabla-$*semipotent ring for every projective module* $P \in mod - R$.

(4) E_F *is a semipotent ring for every free module* $F \in mod - R$.

(5) E_F *is a* $\nabla-$*semipotent ring for every free module* $F \in mod - R$.

Proof. By [6, Theorem 4.10] since for any projective module $P \in mod - R$; $J(E_P) = \nabla E_P$, *by [5, Proposition 1.1]. (See also, [1, Theorem 3.8]).*

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