$I-$Semipotency and the Total of Modules

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ABSTRACT

The object of this paper is to study the total as substructure of $hom_R(M,N)$ ***for any two modules*** M_R ***and*** N_R***, one of interesting question, is when the total of a module*** N ***equals the*** $hom_R(N,J(N))$***. Toward this question, many results have been obtained, where we characterize the module*** N ***and for which*** $\mathrm{Tot}[M,N]=I[M,N]$ ***for all*** $M\in mod-R$***. The main obtained results include:***

(a) ***Let*** N_R ***be module. The following conditions are equivalent:***

(1) ***The module*** N_R ***is an*** $I-$***module.***

(2) $\mathrm{Tot}[M,N]=I[M,N]$ ***for all*** $M\in mod-R$.

(3) $\mathrm{Tot}[N,M]=I[N,M]$ ***for all*** $M\in mod-R$.

(4) $[N,M]$ ***is a*** $I-$***semipotent for all*** $M\in mod-R$.

(5) $[M,N]$ ***is a*** $I-$***semipotent for all*** $M\in mod-R$.

(6) E_N ***is an*** $I-$***semipotent ring and for all*** $M\in mod-R$,

$I\,[M,N]=\{a : a\in[M,N];\ a\,b\in I(E_N)\ \text{for any}\ b\in[N,M]\}$

(b) ***Let*** N_R ***be an*** $I-$***module with*** $\Gamma(N)=\{0\}$***. The following conditions are equivalent:***

(1) E_N ***is a semipotent ring.***

(2) $J(E_N)=I(E_N)$.

(3) ***For every*** $a\in E_N\setminus J(E_N)$ ***there exists a projective direct summand*** $0\neq W\subseteq^{\oplus} N$ ***such that*** $W\subseteq Im(a)$.

Key Words*:* *Semipotent Rings, The total, Jacobson radical, (co) singular ideal, Endomorphisms rings,* $hom_R(M,N)$.

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$I-$ شبه الجمودية وتوتال المودولات

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الملخص

الهدف من هذا العمل هو دراسة التوتال بالنظر إليه كبنية جزئية من المـــودول $hom_R(M,N)$ وذلك لأجل أي مودولين M_R و N_R . أحد الأسئلة المطروحـــة هـــو متـــى يكـــون التوتـــال يـــساوي $hom_R(N,J(N))$ ، أي متى يكون $\text{Tot}(E_N)=hom_R(N,J(N))$ حيث E_N هي حلقة التشاكلات للمودول N_R . في معرض الإجابة عن هذا التساؤل تم الحصول على عدد من النتائج، حيـــث وُصف المودول N_R الذي من أجله $\text{Tot}[M,N]=I[M,N]$ وذلك أياً كان $M\in mod-R$.

النتائج الرئيسة التي تم التوصل إليها من خلال هذه الدراسة هي:

(أ) ليكن N_R مودولاً فوق الحلقة R . الشروط الآتية متكافئة:

1. **المودول N_R هو $I-$ مودول.**
2. **لأجل أي مودول $M\in mod-R$ فإن $\text{Tot}[M,N]=I[M,N]$.**
3. **لأجل أي مودول $M\in mod-R$ فإن $\text{Tot}[N,M]=I[N,M]$.**
4. **لأجل أي مودول $M\in mod-R$ المودول الثنائي $[N,M]$ يكون $I-$ شبه جامد.**
5. **لأجل أي مودول $M\in mod-R$ المودول الثنائي $[M,N]$ يكون $I-$ شبه جامد.**
6. **الحلقة E_N هي حلقة $I-$شبه جامدة وأنه لأجل أي مودول $M\in mod-R$ فإن**

$$I\,[M,N]=\{a : a\in[M,N];\, a\,b\in I(E_N)\ \ for\ any\ b\in[N,M]\}$$

(ب) ليكن N_R عبارة عن $I-$مودول فوق الحلقة R يحقق $\Gamma(N)=\{0\}$. الشروط الآتية متكافئة:

1. **الحلقة E_N هي حلقة شبه جامدة.**
2. $J(E_N)=I(E_N)$.
3. **لأجل أي $a\in E_N\setminus J(E_N)$ يوجد حد مباشر إسقاطي N $0\neq W\subseteq^{\oplus}$ يحقق $W\subseteq Im(a)$.**

الكلمات المفتاحية: الحلقات شبه الجامدة، التوتال، أساس جاكبسون، المثالي المنفـــرد، حلقة التشاكلات لمودول، $hom_R(M,N)$.

رقم التصنيف العالمي للعام 2010: $16E50, 16E60, 16D70$.

1. Introduction.

In this paper rings R are associative with identity unless otherwise indicated. All modules over a ring R are unitary right modules. A submodule N of a module M is said to be small in M if $N + K \neq M$ for any proper submodule K of M, [4]. A submodule N of a module M is said to be large (essential) in M if $N \cap K \neq 0$ for any nonzero submodule K of M, [4]. If M is an $R-$module, the radical of M denoted by $J(M)$ is defined to be the intersection of all maximal submodules of M. Also, $J(M)$ coincides with the sum of all small submodules of M. It my happen that M has no maximal submodules in which case $J(M) = M$, [6]. Thus, for a ring R, $J(R)$ is the Jacobson radical of R. Also, we write $U(R)$ the group of units of a ring R. For a submodule N of a module M, we use $N \subseteq^{\oplus} M$ to mean that N is a direct summand of M, and we write $N \leq_e M$ and $N << M$ to indicate that N is a large, respectively small, submodule of M.

If M_R is a module, we use the notation $E_M = End_R(M)$ and we write $I(E_M) = \{a : a \in E_M ; Im(a) \subseteq J(M)\}$. It is well known that $I(E_M)$ is an ideal in E_M [4]. If M_R and N_R are modules, we use $[M,N] =$ $hom_R(M,N)$. Thus, $[M,N]$ is an $(E_M, E_N)-$bimodule. Our main concern is about the substructures of $hom_R(M,N)$ and the semipotent of $hom_R(M,N)$ (see [7]).

The bimodule $[M,N]$ has there radicals: the Jacobson radical of $[M,N]_{E_M}$, Jacobson radical of ${}_{E_N}[M,N]$, and then the important Jacobson radical denoted $J[M,N]$.

- *The Jacobson radical*

  $J[M,N] = \{a : a \in [M,N];\ b\,a \in J(E_M) \text{ for all } b \in [N,M]\}$

  $J[M,N] = \{a : a \in [M,N];\ a\,b \in J(E_N) \text{ for all } b \in [N,M]\}$

Thus, $J[M,M] = J(E_M)$. In particular, $J[R,R] = J(E_R) \approx J(R)$, hence $E_R \approx R$.

- *The total*

$\mathrm{Tot}[M,N]=\{a : a\in[M,N];\ a\,[N,M]\ \textit{contains no nonzero idempotents}\}$

Where $a\,[N,M]=\{a\,b : b\in[N,M]\}$.

$\mathrm{Tot}[M,N]=\{a : a\in[M,N];[N,M]a\ \textit{contains no nonzero idempotents}\}$

Where $[N,M]a=\{b\,a : b\in[N,M]\}$.

Thus,

$\mathrm{Tot}[M,M]=\mathrm{Tot}(E_M)=\{a : a\in E_M;\ a\,E_M\ \textit{contains no nonzero idempotents}\}$

$=\{a : a\in E_M;\ E_M\,a\ \textit{contains no nonzero idempotents}\}$.

2. $I-$ Semipotent Modules.

Let M_R, N_R *be modules. We put*

$$I[M,N]=\{a : a\in[M,N]; Im(a)\subseteq J(N)\}$$

Thus,

$$I(E_M)=I[M,M]=\{a : a\in E_M; Im(a)\subseteq J(M)\}$$

It is easy to see that

$$I(E_M)=\{a : a\in E_M; b\,a\in I(E_M)\ \textit{for all}\ b\in E_M\}$$

$$I(E_M)=\{a : a\in E_M; a\,b\in I(E_M)\ \textit{for all}\ b\in E_M\}$$

Let P_R *be a projective module then by [6, Proposition 1.1] we have :*

1 - $J(E_P)\subseteq\{a : a\in E_P; Im(a)\subseteq J(P)\}$.

2 – If $J(P)<<P$ *then*

$J(E_P)=\{a : a\in E_P; Im(a)<<J(P)\}=\{a : a\in E_P; Im(a)\subseteq J(P)\}$

.

So, for any projective module P_R *we have:*

1 - $J(E_P)\subseteq I(E_P)$.

2 – If $J(P)<<P$ *then* $J(E_P)=I(E_P)$.

Thus, for a ring R, $I(R)=I[R,R]=J[R,R]=J(E_R)\approx J(R)$, *hence* R_R *is a projective module and* $J(R)<<R_R$.

Also, if M_R *be module and* $K\subseteq^{\oplus} M$, *then* $K\subseteq J(M)$ *if and only if* $J(K)=K\cap J(M)$. *Putting* $\Gamma(M)=\{K : K\subseteq^{\oplus} M;\, J(K)=K\}$. *Not that for any projective module* P, $\Gamma(P)=\{0\}$, *hence if* $K\subseteq^{\oplus} P$ *and* $K\subseteq^{\oplus} J(P)$ *then* K *is projective and* $J(K)=K\cap J(M)=K$, *so* $K=0$. *In addition to, if* $J(M)<<M$ *(or* M *is finitely generated) for some* $M\in mod-R$ *then* $\Gamma(M)=\{0\}$, *hence if* $K\subseteq^{\oplus} M$ *and* $K\subseteq^{\oplus} J(M)$ *then* $K<<M$ *and* $M=K\oplus D$ *for some submodule* D *of* M, *since* $K<<M$ *then* $M=K\oplus D=D$ *and* $0=K\cap D=K\cap M=K$.

We present in the following, a number of results from [3] which will be used throughout the paper.

Lemma 2.1. *[3, Lemma 4.1]. Let* M_R, N_R *be modules then:*
(1) $\mathrm{Tot}[M,N]=\{a : a\in[M,N];\, b\,a\in \mathrm{Tot}(E_M)$ *for all* $b\in[N,M]\}$.
(2) $\mathrm{Tot}[M,N]=\{a : a\in[M,N];\, a\,b\in \mathrm{Tot}(E_N)$ *for all* $b\in[N,M]\}$.

Lemma 2.2. *[3, Lemma 5.10]. Let* M_R, N_R *be modules. The following are equivalent:*

(1) *If* $a\in[M,N]\setminus I[M,N]$ *there exists* $b\in[N,M]$ *such that* $0\neq ba=(ba)^2\in E_M$ *and* $ba\notin I(E_M)$.

(2) *If* $a\in[M,N]\setminus I[M,N]$ *there exists* $b\in[N,M]$ *such that* $0\neq ab=(ab)^2\in E_N$ *and* $a\,b\notin I(E_N)$.

(3) *If* $a\in[M,N]\setminus I[M,N]$ *there exists* $g\in[N,M]$ *such that* $g\,a\,g=g\notin I[N,M]$.

We call that $[M,N]$ *is* $I-$*semipotent [3], if the conditions in lemma 2.2 are satisfied. In particular,* E_M *is an* $I-$*semipotent ring if* *and only if, for any* $a \in E_M$, $a \notin I(E_M)$ *there exists* $b \in E_M$ *such that* $0 \neq ab = (ab)^2 \in E_M$.

Lemma 2.3. *[3, Theorem 5.11]. Let* M_R, N_R *be modules then the following hold:*

(1) *If* $\Gamma(M) = \{0\}$ *then* $[M, N]$ *is* $I-$*semipotent if and only if,* $\mathrm{Tot}[M, N] = I[M, N]$.

(2) *If* $\Gamma(N) = \{0\}$ *then* $[M, N]$ *is* $I-$*semipotent if and only if,* $\mathrm{Tot}[M, N] = I[M, N]$.

In particular, if $\Gamma(M) = \{0\}$ *then* E_M *is an* $I-$*semipotent ring if and only if,* $\mathrm{Tot}(E_M) = I(E_M)$.

Proposition 2.4. *Let* M_R *be module with* $\Gamma(M) = \{0\}$. *The following conditions are equivalent:*

(1) $[M, N]$ *is* $I-$*semipotent for any module* $N \in mod - R$.

(2) E_M *is an* $I-$*semipotent ring and for any module* $N \in mod - R$,

$$I\,[M, N] = \{a : a \in [M, N];\, ba \in I(E_M) \text{ for any } b \in [N, M]\}.$$

Proof. $(1) \Rightarrow (2)$. *Since for any module* $N \in mod - R$, $[M, N]$ *is I- semipotent then by lemma 2.3,* $\mathrm{Tot}[M, N] = I\,[M, N]$, *by[3, Proposition 5.9(b)] we have* $I\,[M, N] = \{a : a \in [M, N];\, b\,a \in I(E_M)$ *for all* $b \in [N, M]\}$. *On the other hand, we have* $\mathrm{Tot}\,(E_M) = \mathrm{Tot}[M, M] = I\,[M, M] = I\,(E_M)$, *so by lemma 2.3* E_M *is an* $I-$*semipotent ring.*

$(2) \Rightarrow (1)$. *Let* N_R *be a module and* $a \in I\,[M, N]$. *Suppose* $a \notin \mathrm{Tot}[M, N]$ *then* $0 \neq (b\,a)^2 = b\,a \in E_M$ *for some* $b \in [N, M]$, *so* $Im\,(b\,a) \oplus Ker\,(b\,a) = M$. *On the other hand, since* $a \in I\,[M, N]$ *then* $Im(a) \subseteq J(N)$ *and* $Im(b\,a) \subseteq J(M)$, *thus* $Im(b\,a) \in \Gamma(M)$, *so* $Im(ba)=0$ *and* $ba=0$ *a contradiction. Hence* $I[M,N]\subseteq \mathrm{Tot}[M,N]$.

Let $a\in \mathrm{Tot}[M,N]$ *then by Lemma 2.1,* $ba\in \mathrm{Tot}(E_M)$ *for all* $b\in[N,M]$. *By (2) since* E_M *is an* $I-$*semipotent ring then by lemma 2.3,* $\mathrm{Tot}(E_M)=I(E_M)$, *so* $ba\in I(E_M)$ *for all* $b\in[N,M]$ *thus, by assumption* $a\in I[M,N]$. *So* $\mathrm{Tot}[M,N]\subseteq I[M,N]$. *Thus we have* $\mathrm{Tot}[M,N]=I[M,N]$ *by lemma 2.3* $[M,N]$ *is* $I-$*semipotent for any module* $N\in mod-R$.

Proposition 2.5. *Let* N_R *be module with* $\Gamma(N)=\{0\}$. *The following conditions are equivalent:*

(1) $[M,N]$ *is* $I-$*semipotent for any module* $M\in mod-R$.

(2) E_N *is an* $I-$*semipotent ring and for any module* $M\in mod-R$,

$$I[M,N]=\{a : a\in[M,N]; ab\in I(E_N) \text{ for all } b\in[N,M]\}.$$

Proof. $(1)\Rightarrow(2)$. *Since for any module* $M\in mod-R$, $[M,N]$ *is I- semipotent then by lemma 2.3,* $\mathrm{Tot}[M,N]=I[M,N]$, *by [3, Proposition 5.9(b)] we have* $I[M,N]=\{a : a\in[M,N]; ab\in I(E_N)$ *for all* $b\in[N,M]\}$. *On the other hand, we have* $\mathrm{Tot}(E_N)=\mathrm{Tot}[N,N]=I[N,N]=I(E_N)$, *so by lemma 2.3* E_N *is an* $I-$*semipotent ring.*

$(2)\Rightarrow(1)$. *Let* M_R *be a module and* $a\in I[M,N]$. *Suppose* $a\notin \mathrm{Tot}[M,N]$ *then* $0\neq(ab)^2=ab\in E_N$ *for some* $b\in[N,M]$, *so* $Im(ab)\oplus Ker(ab)=N$. *On the other hand, since* $a\in I[M,N]$ *then* $Im(ab)\subseteq J(N)$ *so* $Im(ab)\in\Gamma(N)$, *thus* $Im(ab)=0$ *and* $ab=0$ *a contradiction. Hence* $I[M,N]\subseteq \mathrm{Tot}[M,N]$.

Let $a\in \mathrm{Tot}[M,N]$ *then by Lemma 2.1,* $ab\in \mathrm{Tot}(E_N)$ *for all* $b\in[N,M]$. *By (2) since* E_N *is a* $I-$*semipotent ring then by lemma 2.3,* $\mathrm{Tot}(E_N)=I(E_N)$, *so* $ab\in I(E_N)$ *for all* $b\in[N,M]$ *thus, by* *assumption* $a \in I\,[M,N]$. *So* $\mathrm{Tot}[M,N] \subseteq I\,[M,N]$. *Thus we have* $\mathrm{Tot}[M,N] = I\,[M,N]$ *by lemma 2.3* $[M,N]$ *is* $I-$*semipotent for any module* $M \in mod-R$.

Corollary 2.6. *The following conditions are equivalent:*

(1) *For any module* $M \in mod-R$, $\Gamma(M) = \{0\}$ *and* $[M,N]$ *is* $I-$*semipotent for all* $N \in mod-R$.

(2) *For any module* $N \in mod-R$, $\Gamma(N) = \{0\}$ *and* $[M,N]$ *is* $I-$*semipotent for all* $M \in mod-R$.

Proof. $(1) \Rightarrow (2)$. *Let* N_R *be module. Suppose* $M \in mod-R$, *then by (1)* $\Gamma(M) = \{0\}$ *and* $[M,N]$ *is* $I-$*semipotent, and by lemma 2.3 we have* $\mathrm{Tot}[M,N] = I\,[M,N]$. *So by [3, proposition 5.9 (b)] we have*

$$I\,[M,N] = \{a : a \in [M,N];\ a\,b \in I\,(E_N) \text{ for all } b \in [N,M]\}$$

And E_N *is a* $I-$*semipotent ring.*

$(2) \Rightarrow (1)$. *Let* M_R *be module. Suppose* $N \in mod-R$, *then by (2)* $\Gamma(N) = \{0\}$ *and* $[M,N]$ *is* $I-$*semipotent, and by lemma 2.3 we have* $\mathrm{Tot}[M,N] = I\,[M,N]$. *So by [3, proposition 5.9 (b)] we have*

$$I\,[M,N] = \{a : a \in [M,N];\ b\,a \in I\,(E_M) \text{ for any } b \in [N,M]\}$$

And E_M *is a* $I-$*semipotent ring.*

Recall that a ring R *is a semipotent ring [2] if every principal left (resp. right) ideal not contained in* $J(R)$ *contains a nonzero idempotent element, or equivalently, for any* $a \in R$, $a \notin J(R)$ *there exists* $0 \neq x \in R$ *such that* $x = xax$.

A module N_R *is called* $I-$*module if for every submodule* $K \subseteq N$ *such that* $K \not\subset J(N)$ *there exists a projective direct summand* $0 \neq W \subseteq^{\oplus} N$ *with* $W \subseteq K$. *Recall that a module* P_R *is locally projective [5] if for every submodule* $B \subseteq P$, *which is not small in* P *there exists a projective direct summand* $0 \neq W \subseteq^{\oplus} P$ *with* $W \subseteq B$.

In view of definitions $I-$*module, locally projective module implies that a module* N_R *is locally projective if and only if* N_R *is an* $I-$*module and* $J(N) << N$. *Also, we derive the following,*

Lemma 2.7. *Let* P_R *be a projective module. The following conditions are equivalent:*

(1) E_P *is a semipotent rings.*

(2) E_P *is an* $I-$*semipotent rings and* $J(P) << P$.

Lemma 2.8. *Let* N_R *be an* $I-$*module. The following hold:*

(1) $J(E_N) \subseteq I(E_N)$.

(2) *If* $\Gamma(N) = \{0\}$ *then for any module* $M \in mod-R$, $I[M,N] \subseteq \text{Tot}[M,N]$.

Proof. (1). *Let* $a \in J(E_N)$, *suppose that* $a \notin I(E_N)$. *Since* $Im(a) \not\subset J(N)$ *there exists a projective direct summand* $0 \neq W \subseteq^{\oplus} N$ *with* $W \subseteq Im(a)$. *Let* $b : N \to W$ *be the projection of* N *onto* W *then* $0 \neq b = b^2 \in E_N$ *and* $Im(ba) = Im(b) = W \subseteq^{\oplus} N$. *Since* W *is projective then* $Ker(ba) \subseteq^{\oplus} N$, *by [6, Lemma 3.1] there exists* $g \in E_N$ *such that* $(ba)g(ba) = ba$. *Thus,* $0 \neq g(ba) = [g(ba)]^2 \in J(E_N)$, *a contradiction. So* $a \in I(E_N)$.

(2). *Let* $a \in I[M,N]$, *suppose that* $a \notin \text{Tot}[M,N]$ *then there exists* $b \in [N,M]$ *such that* $0 \neq ab = (ab)^2 \in E_N$. *Since* $Im(a) \subseteq J(N)$ *then* $Im(ab) \subseteq J(N)$ *and* $Im(ab) \subseteq^{\oplus} N$, *so* $Im(ab) \in \Gamma(N) = \{0\}$ *therefore* $ab = 0$, *a contradiction.*

Theorem 2.9. *Let* N_R *be an* $I-$*module with* $\Gamma(N) = \{0\}$. *The following conditions are equivalent:*

(1) E_N *is a semipotent ring.*

(2) $J(E_N)=I(E_N)$.

(3) *For every* $a\in E_N\setminus J(E_N)$ *there exists a projective direct summand* $0\neq W\subseteq^{\oplus} N$ *such that* $W\subseteq Im(a)$.

Proof. $(1)\Rightarrow(2)$. *Since* N *is an* $I-$*module then by lemma 2.8* $J(E_N)\subseteq I(E_N)$. *Let* $a\in I(E_N), a\notin J(E_N)$ *by assumption there exists* $b\in E_N$ *such that* $0\neq a\,b=(a\,b)^2\in E_N$. *So* $Im(a\,b)\subseteq^{\oplus} N$ *and* $Im(a\,b)\subseteq Im(a)\subseteq J(N)$ *thus,* $Im(a\,b)\in\Gamma(N)=\{0\}$ *therefore* $a\,b=0$, *a contradiction so* $a\in J(E_N)$.

$(2)\Rightarrow(3)$. *Suppose that* $J(E_N)=I(E_N)$. *Let* $a\in E_N\setminus J(E_N)$ *then* $a\notin I(E_N)$ *and*

$Im(a)\not\subset J(N)$, *since* N *is an* $I-$*module* $a\in E_N\setminus J(E_N)$ *there exists a projective direct summand* $0\neq W\subseteq^{\oplus} N$ *such that* $W\subseteq Im(a)$.

$(3)\Rightarrow(1)$. *Let* $g\in E_N\setminus J(E_N)$ *then there exists a projective direct summand* $0\neq V\subseteq^{\oplus} N$ *such that* $V\subseteq Im(g)$. *Let* $p:N\to V$ *be the projection of* N *onto* V, *then* $Im(p\,g)=Im(g)=V$ *hence* $Im(p)=V\subseteq Im(g)$. *Since* V *is projective then* $Ker(p\,g)\subseteq^{\oplus} N$ *by [2, Theorem 2.2] follows that* E_N *is a semipotent ring.*

Theorem 2.10. *Let* N_R *be module with* $\Gamma(N)=\{0\}$. *The following conditions are equivalent:*

(1) *The module* N_R *is an* $I-$*module.*

(2) $[M,N]$ *is an* $I-$*semipotent for all* $M\in mod-R$.

(3) $\mathrm{Tot}[M,N]=I\,[M,N]$ *for all* $M\in mod-R$.

Proof. $(1)\Rightarrow(3)$. *Let* $a\in\mathrm{Tot}[M,N]$, *suppose that* $a\notin I\,[M,N]$ *then* $Im(a)\not\subset J(N)$ *by assumption there exists a projective direct summand* $0\neq P\subseteq^{\oplus} N$ *with* $P\subseteq Im(a)$. *By* *modular law* $P \subseteq^{\oplus} Im(a)$. *Let* $p : Im(a) \to P$ *the projection, then* $pa : M \to P$ *is epimorphism. Since* P *is projective then* $M = A \oplus Ker\,(pa)$ *for some submodule,* $A \subseteq M$ *and* $pa : A \to P$ *is isomorphism. By [5, Lemma 1.1] there exists* $g \in [N, M]$ *such that* $0 \neq g(pa) = g(pa)^2 \in E_M$, *and by [5, Lemma 1.2] there exists* $b \in [N, M]$ *such that* $0 \neq ba = (ba)^2 \in E_M$, *a contradiction.*

$(3) \Rightarrow (1)$. *Let* B *be a submodule of* N *and* $B \not\subset J(N)$. *By [1, Theorem 2.3], denote by* $l : P \to B$ *a projective extension of* B *and by* $t : B \to N$ *the inclusion. Let* $g = tl : P \to N$ *then* $Im(g) = B \not\subset J(N)$, *so* $g \in [P, N] \setminus I[P, N]$ *by assumption there exists* $m \in [N, P]$ *such that* $0 \neq mg = (mg)^2 \in E_P$, *by [5, Lemma 1.1] there exists* $0 \neq P_0 \subseteq^{\oplus} P$ *and* $0 \neq N_0 \subseteq^{\oplus} N$ *such that* $g : P_0 \to N_0$ *is isomorphism. Since* P_0 *is projective then* N_0 *is projective and* $N_0 = g(P_0) \subseteq Im(g) = B$. *Thus,* N *is an* $I-$*module.* $(2) \Leftrightarrow (3)$ *by lemma 2.3.*

Proposition 2.11. *Let* N_R *be an* $I-$*module with* $\Gamma(N) = \{0\}$. *Then for any module* $M \in mod - R$, *the following conditions are equivalent:*

(1) $[M, N]$ *is semipotent.*

(2) $\mathrm{Tot}[M, N] = J[M, N] = I\,[M, N]$.

(3) *For any* $a \in [M, N] \setminus J[M, N]$ *there exists* $b \in [N, M]$ *such that* $0 \neq Im\,(ab) \subseteq^{\oplus} N$.

Proof. $(1) \Rightarrow (2)$. *Since* $[M, N]$ *is semipotent then By [7, Theorem 2.2]* $\mathrm{Tot}[M, N] = J[M, N]$ *and by theorem 2.10 we have* $\mathrm{Tot}[M, N] = I\,[M, N]$ *hence,* N_R *is an* $I-$ *module.*

$(2) \Rightarrow (1)$. *By theorem 2.10 and [7, Theorem 2.2].*

$(2) \Rightarrow (3)$. *Let* $a \in [M, N] \setminus J[M, N]$ *then* $a \notin \mathrm{Tot}[M, N]$ *so there exists* $b \in [N, M]$ *such that* $0 \neq ab = (ab)^2 \in E_N$ *thus* $Im(ab) \subseteq^{\oplus} N$.

$(3) \Rightarrow (2)$. *Since* N_R *is an* $I-$*module then by theorem 2.10,* $\text{Tot}[M,N] = I\,[M,N]$ *for every module* $M \in mod-R$. *On the other hand, we have* $J\,[M,N] \subseteq \text{Tot}\,[M,N]$. *Let* $a \in \text{Tot}\,[M,N]$, *suppose* $a \notin J\,[M,N]$ *by assumption there exists* $b \in [N,M]$ *such that* $0 \neq Im\,(a\,b) \subseteq^{\oplus} N$. *Since* $a \in \text{Tot}\,[M,N]$ *then* $a \in I\,[M,N]$ *and* $Im(a) \subseteq J(N)$, *thus* $Im\,(a\,b) \in \Gamma(N)$ *hence* $Im\,(a\,b) \subseteq Im(a)$, *which follows that* $a\,b = 0$ *a contradiction.*

Corollary 2.12. *Let* N_R *be module with* $\Gamma(N) = \{0\}$. *The following conditions are equivalent:*

(1) *The module* N_R *is an* $I-$*module.*

(2) $\text{Tot}[M,N] = I\,[M,N]$ *for all* $M \in mod-R$.

(3) $\text{Tot}[N,M] = I\,[N,M]$ *for all* $M \in mod-R$.

(4) $[N,M]$ *is an* $I-$*semipotent for all* $M \in mod-R$.

(5) $[M,N]$ *is an* $I-$*semipotent for all* $M \in mod-R$.

(6) E_N *is an* $I-$*semipotent ring and for any module* $M \in mod-R$,

$$I\,[M,N] = \{a : a \in [M,N];\, a\,b \in I\,(E_N) \text{ for any } b \in [N,M]\}$$

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (5)$ *by theorem 2.10.* $(2) \Leftrightarrow (3)$ *by corollary 2.6. and* $(3) \Leftrightarrow (4)$ *by lemma 2.3.* $(2) \Leftrightarrow (6)$ *by lemma 2.3 and proposition 2.5.*

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