On Upper Fuzzy Prime Ideal over Rings

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ABSTRACT

In this paper we shall study the definition of upper fuzzy prime ideals, T-upper fuzzy prime ideals and T-S- upper weakly fuzzy prime ideals proving the inclusion relationships that are satisfied among them. Examples are given showing that some relationships don't hold between these types of ideals. On the other hand we use these definitions to study some properties, proposition and theorems.

Key Words: Fuzzy Sets; Upper fuzzy subrings; Upper Fuzzy Ideals; Upper Fuzzy Prime deals; Level Upper Fuzzy Subrings; T- Upper Fuzzy Prime Ideals; T-S- Upper Weakly Fuzzy Prime Ideals.

Mathematical subjects classification (MSC 2010): $16D_{25}$

المثاليات الأولية المشوشة العليا على الحلقات

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الملخص

دُرستْ في هذه الورقة البحثية كلٌّ من المثاليات الأولية المشوشة العليا و T- المثاليات الأولية المشوشة العليا وكذلك S-T- المثاليات الأولية الضعيفة المشوشة العليا وذلك من خلال العلاقة التي تربط فيما بينها، كما طُرحتْ بعض الأمثلة التي تبين عدم تحقق بعض العلاقات بين هذه الأنواع من المثاليات. من جهة أخرى اِستُخدمت المفاهيم السابقة لدراسة بعض الخواص والقضايا والمبرهنات.

الكلمات المفتاحية: المجموعات المشوشة، الحلقات الجزئية المشوشة العليا، المثاليات المشوشة العليا، المثاليات الأولية المشوشة العليا، سوية الحلقات الجزئية المشوشة العليا، T– المثاليات الأولية المشوشة العليا، S-T- المثاليات الأولية الضعيفة المشوشة العليا.

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1. Introduction:

The concept of a fuzzy subset was introduced by Zadeh (1965)[13]. Fuzzy subgroups and its important properties were defined and established by Rosenfeld (1971) [11]. After this time it was necessary to define a fuzzy ideal of ring. The notion of a fuzzy ideal was introduced by Liu. Malik and Mordeson and Mukherjee [7, 8] and have studied fuzzy ideals. Dixit and N.Ajmal [4] studied some definitions of fuzzy prime ideals. N.Palaniappan and Jin [9,5,6] defined fuzzy prime ideal of a Γ- ring also T.K.Dutta and Tanusree Chanda [2]. On the other hand Xiang and Jian [12] studied the fuzzy prime ideal of ordered semi groups. P. Dheena, Syam Prasad and Bhavanari [1,10] discussed this subject. In this paper, we shall study the upper fuzzy prime ideal and T-S- upper weakly fuzzy prime ideal for definition, example, some properties and theorems.

Throughout this paper, R stands for a commutative ring, Z is the integer set, <x> is a main ideal generated by the element x, Im(μ) is the image of μ, χ_A is the characteristic of a set A and min, max denoted to minimum and maximum respectively.

2. Preliminary Definitions:

Definition 2.1 [13] Let X be a non empty set, a fuzzy subset μ of X is just a function from X onto [0,1].

Definition 2.2 [13] Let δ, μ be two fuzzy subsets of R, δ subset of μ (symbol $\delta \subseteq \mu$) if

$\delta(x) \leq \mu(x)$ for all $x \in R$.

Definition 2.3 [8] Let δ, μ be two fuzzy subsets of R, then $\delta \circ \mu$ defined by

$$\delta \circ \mu (x) = \begin{cases} \bigvee\limits_{x=yz}\big(\delta(y) \wedge \mu(z)\big) & ; \text{ if } x \text{ is expressible as } x = yz \\ 0 & ; \text{ if } x \text{ is not expressible as } x = yz \end{cases}$$

for all $x \in R$

Defintion 2.4 [8] Let R be a ring and μ be a fuzzy subset of R, then μ is said to be an upper fuzzy left (right) ideal over R if

(i) $\mu(x - y) \leq \max\{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \leq \mu(y)$ ($\mu(xy) \leq \mu(x)$) for all $x, y \in R$.

Remark 2.5 If μ is an upper fuzzy left and right ideal of R, i.e

(i) $\mu(x - y) \leq \max\{\mu(x), \mu(y)\}$; $\forall x, y \in R$.

(ii) $\mu(xy) \leq \min\{\mu(x), \mu(y)\}$; $\forall x, y \in R$.

Then μ is said to be an upper fuzzy ideal of R .

Proposition 2.6 [10] Let I be an ideal of R and $a, b \in [0, 1]$ such that $a \leq b \neq 0$, then μ which is defined by:

$$\mu(x) = \begin{cases} a & ; x \in I \\ b & ; x \in R \setminus I \end{cases}$$

Is an upper fuzzy ideal of R.

Definition 2.7[8] Let R be a ring and let μ be an upper fuzzy subring of R. The subset μ_t such that $t \in [0, 1]$ is called the level upper fuzzy subring of μ and defined by $\mu_t = \{x \in R, \mu(x) \geq t\} = \mu^{-1}([t, 1])$.

Definition 2.8[9] Let R be a ring and let S be a subset of R such that $S \neq \phi$. S is called a completely prime subset of R if for all x, y in R with $x.y \in S$ then $x \in S$ or $y \in S$.

Definition 2.9[1] A mapping $T: [0,1] \times [0,1] \to [0,1]$ is called a triangular norm iff it satisfies the following conditions

1- $T\{a, 1\} = T\{1, a\} = a$; $\forall a \in [0, 1]$

2- If $a \geq b, c \geq d$ then $T\{a, c\} \geq T\{b, d\}$; $\forall a, b, c, d \in [0, 1]$

3- $T\{a, b\} = T\{b, a\}$; $\forall a, b \in [0, 1]$

4- $T\{a, T\{c, d\}\} = T\{T\{a, c\}, d\}$; $\forall a, c, d \in [0, 1]$.

Definition 2.10[6] A mapping $S: [0,1] \times [0,1] \to [0,1]$ is called a triangular conorm iff it satisfies the following conditions.

1- $S\{0, a\} = S\{a, 0\} = a$; $\forall a \in [0, 1]$

2- If $a \geq b, c \geq d$ then $S\{a, c\} \geq S\{b, d\}$; $\forall a, b, c, d \in [0, 1]$

3- $S\{a,b\} = S\{b,a\};\ \forall a,b \in [0,1]$

4- $S\{a,S\{c,d\}\} = S\{S\{a,c\},d\};\forall a,c,d \in [0,1]$.

Remark 2.11 If $T\{x,y\} = \min\{x,y\}$ then T is called **min norm**, also if $S\{x,y\} = \max\{x,y\}$ then S is called **max conorm**.

3. Upper Fuzzy Prime Ideals

Definition 3.1 An upper fuzzy ideal μ of R is said to be an upper fuzzy prime ideal of R, if μ is not a constant function and for any upper fuzzy ideals δ, τ in R. If $\delta \circ \tau \subseteq \mu$, then either $\delta \subseteq \mu$ or $\tau \subseteq \mu$.

Example 3.2 Let Z be the integer ring and let μ be the upper fuzzy ideal of Z given by

$$\mu(x) = \begin{cases} 0.9\,; x \in Z - <3> \\ 0.5\,; x \in <3> - \{0\} \\ 0.1\,; x = 0 \end{cases}$$

Then μ is an upper fuzzy prime ideal.

Remark 3.3 [12] Let $\mu_0 = \{x \in R\,;\ \mu(x) = \mu(0)\}$ such that μ is a fuzzy subset of R, then:

(1) If μ is an upper fuzzy ideal of R, then μ_0 is an ideal.

(2) If μ is an upper fuzzy prime ideal of R, then μ_0 is a prime ideal.

(3) If μ is an upper fuzzy prime ideal over the ring Z, then $\mu_0 = \{0\}$.

(4) If μ_0 is a prime ideal , then μ is not necessarily an upper fuzzy prime ideal .

Proposition 3.4 Let I be an ideal in R, $n \in [0,1[$, and let μ be a fuzzy subset of R defined by :

$$\forall x \in R: \mu(x) = \begin{cases} n\ ;\ x \in I \\ 1\ ;\text{otherwise} \end{cases}$$

Then μ is an upper fuzzy prime ideal.

Proof:

μ is an upper fuzzy ideal by Proposition 2.6, let λ, δ be any upper fuzzy ideals and suppose that $\lambda \not\subset \mu$, $\delta \not\subset \mu$, then there exist x, y $\in$ R such that

$\lambda(x) > \mu(x)$, $\delta(y) > \mu(y)$ then we get $\mu(x) = n = \mu(y)$ and hence x, y $\in$ I so $x.y \in I$, and we get $\mu(x.y) = n$ also we know that $(\lambda \circ \delta)(x.y) = \min\{\lambda(x), \delta(y)\}$. But $\lambda(x) > \mu(x) = n$, $\delta(y) > \mu(y) = n$ then $\min\{\lambda(x), \delta(y)\} > n = \mu(x.y)$ so that $\lambda \circ \delta \not\subset \mu$ therefore μ is an upper fuzzy prime ideal of R.

Proposition 3.5 [8] If μ is an upper fuzzy prime ideal of a finite ring R, then $|\mathrm{Im}(\mu)| = 2$, this means that μ has two- valued.

Proposition 3.6[9]

(1) If μ is a fuzzy subset of R satisfies that $\mu(0) = 1$ and has two – valued also if μ_0 is a prime ideal of R, then μ is an upper fuzzy prime ideal of R.

(2) I is a prime ideal of R iff χ_I is an upper fuzzy prime ideal of R.

Corollary 3.7 Let R be a commutative ring with unity ,then every non constant upper fuzzy ideal μ of R with $\mu(0) = 1$ is an upper fuzzy prime ideal iff R is a field.

Proof:-

(¬) Suppose that R is a field, then every non constant upper fuzzy ideal of R with

$\mu(0) = 1$ is such that for all x in R; $x \neq 0$, $\mu(x) = c$ where $0 \leq c < 1$. Since $\mu_0 = \{0\}$ is a prime ideal of R but by Proposition 3.6(1) μ is an upper fuzzy prime ideal of R.

(®) If every such μ is an upper fuzzy prime ideal then every ideal I $\neq$ R is prime. If I $\neq$ R is an ideal but not prime, then χ_I is not an upper fuzzy prime ideal of R, therefore R is a field.

Now we remember in the Noetherian ring each strictly increasing chain of ideals is stationary. If a chain is formed by i + 1 ideals it has a length i. The dimension of R is the supreme of the lengths of all the chains of prime ideals of R.

Lemma 3.8 Every upper fuzzy prime ideal of a Noetherian ring R of finite dimension is finitely valued.

Proof: If all the chains of prime ideals of R have finite length , all the chains of level ideals for an upper fuzzy prime ideal μ have a finite number of elements. Thus μ is finitely valued.

Definition 3.9 Let R be a ring, T be a triangular norm and S be a triangular conorm. A fuzzy subset μ of R such that $\mu \neq 0$ is called a T-S-upper fuzzy ideal of R iff it satisfies the following conditions:

1- $\mu(x-y) \geq T\{\mu(x), \mu(y)\}; \forall x, y \in R$
2- $\mu(x) = \mu(-x); \forall x \in R$
3- $\mu(x.y) \geq S\{\mu(x), \mu(y)\}; \forall x, y \in R$.

Remark 3.10

(1) If the triangular conorm S is the max conorm, then the T-max-upper fuzzy ideal is called T- upper fuzzy ideal.
(2) If the triangular norm T is the min norm, then the min-S-upper fuzzy ideal is called S-upper fuzzy ideal.

Definition 3.11 Let λ be a non constant T-upper fuzzy ideal of a commutative ring R. λ is called a T-upper fuzzy prime ideal of R iff $\lambda(x.y) = \max\{\lambda(x), \lambda(y)\}$ for all $x, y \in R$.

Example 3.12 Let Z be the integers ring, T be the product norm and let μ be the fuzzy subset of Z defined by

$$\mu(x) = \begin{cases} 0.2 \ ; x \notin <2> \cup <3> \\ 0.6 \ ; x \in <2> \cup <3> \end{cases}$$

μ is a T- upper fuzzy prime ideal.

Lemma 3.13 Let R be a ring and μ be a non- constant T- upper fuzzy ideal of R, $t \in [0,1]$. Then μ is T- upper fuzzy prime ideal of R iff the level subsets μ_t are completely prime subsets of R.

Proof:

(¬) Let $x, y \in R$, If μ (x .y) = t, then $x.y \in \mu_t$. Now if μ_t is a completely prime subset , then $x \in \mu_t$ or $y \in \mu_t$ therefore $\mu (x) \geq t$ or $\mu (y) \geq t$, hence

$\mu (x.y) = t \leq \max \{\mu (x), \mu (y)\}$ also μ is a T- upper fuzzy ideal of R, thus $\mu (x.y) \geq \max \{\mu (x), \mu (y)\}$ and by above we get $\mu (x.y) = \max \{\mu (x), \mu (y)\}$ so μ is a T-upper fuzzy prime ideal.

(®) Let $x, y \in R$ such that $x.y \in \mu_t$, then $\mu (x.y) \geq t$, we know μ is a T-upper fuzzy prime ideal, therefore $\mu (x.y) = \max \{\mu (x), \mu (y)\} \geq t$. Then $\mu (x) \geq t$ or $\mu (y) \geq t$ it follows that $x \in \mu_t$ or $y \in \mu_t$ and μ_t is a completely prime subset of R.

Definition 3.14 Let μ be a non constant T-S-upper fuzzy ideal of R. μ is called a T-S-upper weakly fuzzy prime ideal of R if for all x, y $\in$ R with $\mu (x.y) = \mu (0)$ then $\mu (x) = \mu (0)$ or $\mu (y) = \mu (0)$.

Remark 3.15

(1) If the triangular conorm S is the max conorm, then the T-max-upper weakly fuzzy prime ideal is called T- upper weakly fuzzy prime ideal.

(2) If the triangular norm T is the min norm, then the min-S- upper weakly fuzzy prime ideal is called S- upper weakly fuzzy prime ideal.

Example 3.16 Let μ be the T- upper fuzzy ideal of the integers ring Z with T the product norm defined by

$$\mu (x) = \begin{cases} 0.9 \; ; x \in Z - < 3 > \\ 0.7 \; ; x \in \{ -3 , 3 \} \\ 0.3 \; ; x \in < 3 > - \{ -3 , 0 , 3 \} \\ 0 \quad ; \; x = 0 \end{cases}$$

μ is a T- upper weakly fuzzy prime ideal.

Theorem 3.17 Let μ be a T-S-upper fuzzy ideal of R, then μ is a T-S- upper weakly fuzzy prime ideal iff $\mu_0 = \{x \in R, \mu(x) = \mu(0)\}$ is a completely prime subset.

Proof:

Let μ be a T-S-upper fuzzy ideal, then μ_0 is a completely prime subset iff for all $x.y \in \mu_0$ then $x \in \mu_0$ or $y \in \mu_0$ iff $\mu(x.y) = \mu(0)$ then $\mu(x) = \mu(0)$ or

$\mu(y) = \mu(0)$ iff μ is a T-S- upper weakly fuzzy prime ideal.

Corollary 3.18 Every T-upper fuzzy prime ideal is a T-upper weakly fuzzy prime ideal.

Proof:

By Lemma 3.13. If μ is a T-upper fuzzy prime ideal, then the level subsets μ_t are completely prime subset of R, where $t \in [0, 1]$. Then μ_0 is a completely prime subset also by Theorem 3.17 μ is a T-upper weakly fuzzy prime ideal.

A T-upper weakly fuzzy prime ideal is not necessarily T-upper fuzzy prime ideal. As the example 3.16 shows.

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