Regular and Semipotent Rings

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Abstract

Let R be a ring with identity.

The ain is to study some fundamental properties of a ring R when R is regular or semi-potent and the radical Jacobson of R when R is semi-potent.

New results were obtained including necessary and sufficient conditions of R to be regular or semi-potent. New substructures of R are studied and their relationship with the total of R.

Key Words: *Regular ring, Semipotent Rings, The total, The Jacobson radical, Annihilator.*

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الحلقات المنتظمة وشبه الجامدة

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الملخص

لتكن R حلقة واحدية.
الهدف من هذه الورقة هو دراسة بعض الخواص الأساسية للحلقة R عندما تكون الحلقة R منتظمة أو شبه جامدة، ودراسة أساس جاكبسون للحلقة R عندما تكون الحلقة R شبه جامدة.
تم الحصول على نتائج جديدة تتضمن عدداً من الشروط اللازمة والكافية كي تكون الحلقة R منتظمة أو شبه جامدة. ودُرستْ بنى جزئية جديدة في الحلقة R فضلاً عن دراسة علاقة هذه البنى الجزئية بالتوتال للحلقةR.

الكلمات المفتاحية: الحلقة المنتظمة، الحلقة شبه الجامدة، أساس جاكبسون، التوتال، العادم.

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1. Introduction

In this paper rings R*, are associative with identity unless otherwise indicated. All modules over a ring* R *are unitary right modules. We write* $J(R)$ *and* $U(R)$ *for the Jacobson radical and the group of units of a ring* R*. For any element* a *of a ring* R*, we write* $r(a)$ *and* $l(a)$ *the right, respectively left, annihilator of* a *in* R*.*

2. Regular Rings.

We start with the following fundamental lemma which gives information about relationship between any two elements of a ring R*.*

Lemma 2.1. *Let* R *be a ring and* $a,b\in R$*. The following hold:*

(1) $R = aR + (1-ab)R$.

(2) $(a-aba)R = aR\cap(1-ab)R$.

(3) $r(a)\cap r(1-ba)=\{0\}$.

(4) $r(a-aba) = r(a)+r(1-ba)$.

$(4+i)$ *the left-right symmetry of* (i)*,* $i=1,2,3,4.$.

Proof. (1) *It is clear, hence* $R = abR+(1-ab)R\subseteq aR+(1-ab)R\subseteq R$.

(2) *It is clear that* $(a-aba)R = a(1-ba)R\subseteq aR$ and
$(a-aba)R=(1-ab)aR\subseteq(1-ab)R$*, so*
$(a-aba)R\subseteq aR\cap(1-ab)R$.

Let $x\in aR\cap(1-ab)R$*. Then* $x = ar_1 = (1-ab)r_2$ *where* $r_1,r_2\in R$ *and that*

$$r_2 = x+abr_2 = ar_1+abr_2 = a(r_1+br_2).$$

For $r_0 = r_1+br_2\in R$

$$x=(1-ab)r_2 = (1-ab)ar_0 = (a-aba)r_0\in(a-aba)R.$$

So $aR \cap (1-ab)R \subseteq (a-aba)R$ *and* $aR \cap (1-ab)R = (a-aba)R$.

(3) *Let* $x \in r(a) \cap r(1-ba)$. *Then* $ax = (1-ba)x = 0$ *and that* $x = bax = 0$.

(4) *It is clear that* $r(a) + r(1-ba) \subseteq r(a-aba)$, *hence if* $x \in r(a)$*;* $ax = 0$ *and* $(a-aba)x = 0$. *Also, if* $y \in r(1-ba)$*;* $(1-ba)y = 0$ *and* $(a-aba)y = 0$.

Let $z \in r(a-aba)$. *Then* $(a-aba)z = 0$ *and that* $az = abaz$. *On the other hand,* $z = baz + (1-ba)z$ *and* $baz \in r(1-ba)$, $(1-ba)z \in r(a)$, *hence*

$$(1-ba)baz = baz - babaz = baz - baz = 0$$

$a(1-ba)z = (a-aba)z = 0$, *so* $r(a-aba) \subseteq r(a) + r(1-ba)$.

Definition: *An element* a *of a ring* R *is called regular [1], if* $a = aba$ *for some* $b \in R$. *The next Lemma gives information about* $a \in R$, *when* a *is regular.*

Lemma 2.2. *Let* R *be a ring and* $a \in R$. *The following are equivalent:*

(1) *There exists* $b \in R$ *such that* $a = aba$.

(2) *There exists* $b \in R$ *such that* $aR \cap (1-ab)R = \{0\}$.

(3) *There exists* $b \in R$ *b such that* $r(a) + r(1-ba) = R$.

$(3+i)$ *The left-right symmetry of* $(1+i)$ *for* $i = 1,2$.

Proof. $(1) \Rightarrow (2)$. *Suppose* (1) *holds. Let* $x \in aR \cap (1-ab)R$. *Then*

$$x = ar_1 = (1-ab)r_2$$

Where $r_1, r_2 \in R$. *So*

$$r_2 = x + abr_2 = ar_1 + abr_2 = a(r_1 + br_2) = ar_0$$

Where $r_0 = r_1 + br_2$.

Thus $x = (1-ab)r_2 = (1-ab)ar_0 = ar_0 - abar_0 = ar_0 - ar_0 = 0$.

By assumption. giving (2).

$(2) \Rightarrow (3)$. *Suppose* (2) *holds. By Lemma 2.1,* $(a-aba)R = aR \cap (1-ab)R = \{0\}$. *Thus,* $a - aba = 0$ *and* $R = r(0) = r(a-aba) = r(a) + r(1-ba)$ *by Lemma 2.1.*

$(3) \Rightarrow (1)$. *Suppose* (3) *holds. By Lemma 2.1,* $r(a-aba) = r(a) + r(1-ba) = R$, *so* $1 \in r(a-aba)$ *and that* $a - aba = 0$.

Definition: *A ring* R *is called regular [4], if and only if for any* $a \in R$, a *is regular. The following Lemma describe the principal left and right ideals of a ring* R *when* R *is regular.*

Lemma 2.3. *Let* R *be a regular ring and* $a, b \in R$. *The following hold:*

(1) $aR = \{x \in R : xR \subseteq aR\}$.

(2) $Ra \subseteq Rb$ *if and only if* $r(b) \subseteq r(a)$.

(3) $Ra = Rb$ *if and only if* $r(b) = r(a)$.

(4) $Ra = \{y \in R :\ r(y) \subseteq r(a)\}$.

Proof. (1) *it is clear.* (2) $(\Rightarrow)$ *obvious.*

$(\Leftarrow)$. *Suppose that* $r(b) \subseteq r(a)$. *Since* R *is regular there exists* $d \in R$ *such that* $b = bdb$. *For* $e = db$; $r(e) = r(b)$ *and that* $1 - e \in r(b) \subseteq r(a)$. *Thus,* $a(1-e) = 0$ *and* $a = ae = adb \in R$. (3) *and* (4) *are obvious by* (2).

3. Semipotent Rings.

Definition: *An element* a *of a ring* R *is called partially invertible or pi for short [2], if* a *is a divisor of an idempotent element. The next Lemma gives information about* $a \in R$, *when* a *is a divisor of an idempotent.*

Lemma 3.1. *Let* R *be a ring and* $a \in R$. *The following are equivalent:*

(1) *There exists* $0 \neq b \in R$ *such that* $b = bab$.

(2) *There exists* $0 \neq b \in R$ *such that* $(ab)^2 = ab$.

(3) *There exists* $0 \neq b \in R$ *such that* $(ba)^2 = ba$.

(4) *There exists* $0 \neq b \in R$ *such that* $bR \cap (1-ba)R = \{0\}$.

(5) *There exists* $0 \neq b \in R$ *such that* $r(b) + r(1-ab) = R$.

$(5+i)$ *The left-right symmetry of* $(3+i)$ *for* $i = 1,2$.

Proof. $(1) \Rightarrow (2)$. *Suppose* (1) *holds. Then* $(ab)^2 = ab$ *gives* (2).
$(2) \Rightarrow (1)$. *Suppose* (2) *holds. Then for* $d = bab$; $d = dad$ *gives* (1). *Similarly, the equivalence* $(1) \Leftrightarrow (3)$ *holds.*
$(1) \Rightarrow (4)$. *Suppose that* $b = bab$ *for some* $0 \neq b \in R$. *Then by Lemma 2.1(2)* $bR \cap (1-ba)R = \{0\}$.
$(4) \Rightarrow (5)$. *Suppose that* $bR \cap (1-ba)R = \{0\}$ *for some* $0 \neq b \in R$. *Then by Lemma 2.1(2);* $b = bab$ *and by Lemma 2.1;* $r(b) + r(1-ab) = R$.
$(5) \Rightarrow (1)$. *Suppose* $r(b) + r(1-ab) = R$ *for some* $0 \neq b \in R$. *Then by Lemma 2.1;* $r(b - bab) = R$. *Since* $1 \in R = r(b - bab)$ *implies* $b - bab = 0$.

Lemma 3.2. *Let* R *be a ring and* $a,b\in R$. *The following hold:*

(1) $(1-ab)R=R$ *if and only if* $(1-ba)R=R$.

(2) $r(1-ab)=\{0\}$ *if and only if* $r(1-ba)=\{0\}$.

(3) $1-ab\in U(R)$ *if and only if* $1-ba\in U(R)$.

Proof. (1) $(\Rightarrow)$. *Suppose that* $(1-ab)R=R$. *Then by Lemma 2.1,*

$$bR\cap(1-ba)R=(b-bab)R=b(1-ab)R=bR$$

so $bR\subseteq(1-ba)R$ *and that* $R=(1-ba)R$. *Similarly* $(\Leftarrow)$ *holds.*

(2) $(\Rightarrow)$. *Suppose that* $r(1-ab)=\{0\}$. *Let* $x\in r(1-ba)$. *Then* $(1-ba)x=0$, *so* $(a-aba)x=0$ *and that* $(1-ab)ax=0$. *Thus,* $ax\in r(1-ab)=\{0\}$, *so* $ax=0$ *and* $x\in r(a)$. *This shows That* $r(1-ba)\subseteq r(a)$ *and that* $r(1-ba)=r(a)\cap r(1-ba)=\{0\}$.

Similarly $(\Leftarrow)$ *holds.*

(3) *by* (1) *and* (2).

Definition: *A non-empty subset* B *of a ring* R *be a ring is called semi-ideal in* R *[2], if for any* $a\in R, b\in B$, *then* $ab, ba\in B$.

Definition: *Let* R *be a ring. Write :*

$$\nabla_1(R)=\{a\in R:\ (1-ab)R=R \text{ for all } b\in R\}$$

$$\nabla_2(R)=\{a\in R:\ (1-ba)R=R \text{ for all } b\in R\}$$

It is clear that $\nabla_1(R)$ *and* $\nabla_2(R)$ *are non-empty subsets in* R *(* $0\in\nabla_1(R)$, $0\in\nabla_2(R)$*). Using Lemma 3.2(1), it is easy to see that* $\nabla_1(R)=\nabla_2(R)$. *Therefore, we use the notation:*

$$\hat{\nabla}(R) = \{a \in R : (1-ab)R = R \quad \textit{for all } b \in R\}$$
$$= \{a \in R : (1-ba)R = R \quad \textit{for all } b \in R\}$$

$\hat{\nabla}(R)$ is a semi-ideal in R, which means that it is closed under arbitrary multiplication from either side, hence if $a \in \hat{\nabla}(R)$ and $d \in R$;

$$(1-b(da))R = (1-(bd)a)R = R$$

for all $b \in R$ and

$$(1-(ad)b)R = (1-a(db))R = R$$

for all $b \in R$. Thus, $ad, da \in \hat{\nabla}(R)$.

Let R be a ring. Write :

$$\Delta_1(R) = \{a \in R : r(1-ab) = \{0\} \textit{ for all } b \in R\}$$
$$\Delta_2(R) = \{a \in R : r(1-ba) = \{0\} \textit{ for all } b \in R\}$$

It is clear that $\Delta_1(R)$ and $\Delta_2(R)$ are non-empty subsets in R ($0 \in \Delta_1(R), 0 \in \Delta_2(R)$). Using Lemma 3.2(2), it is easy to see that $\Delta_1(R) = \Delta_2(R)$. Therefore, we use the notation:

$$\hat{\Delta}(R) = \{a \in R : r(1-ab) = \{0\} \quad \textit{for all } b \in R\}$$
$$= \{a \in R : r(1-ba) = \{0\} \quad \textit{for all } b \in R\}$$

$\hat{\Delta}(R)$ is a semi-ideal in R, which means that it is closed under arbitrary multiplication from either side, hence if $a \in \hat{\Delta}(R)$ and $d \in R$;

$$r(1-b(da)) = r(1-(bd)a) = \{0\}$$

for all $b \in R$ and

$$r(1-(ad)b) = r(1-a(db)) = \{0\}$$

for all $b \in R$. *Thus,* $ad, da \in \hat{\Delta}(R)$.

Following [2], $\text{Tot}\,(R) = \{a \in R:\ a \text{ is not pi}\}$.

Lemma 3.3. *For any ring* R. *The following hold:*

(1) $J(R) \subseteq \hat{\nabla}(R) \cap \hat{\Delta}(R)$.

(2) $\hat{\nabla}(R) \cup \hat{\Delta}(R) \subseteq \text{Tot}\,(R)$.

(3) $J(R) \subseteq \text{Tot}\,(R)$.

Proof. (1). *Let* $a \in J(R)$. *Then for every* $b \in R$; $ba, ab \in J(R)$, *so there exists* $c, d \in R$ *such that* $d(1-ba) = 1$ *and* $(1-ab)c = 1$. *Thus,* $r(1-ba) = \{0\}$ *and that* $R = (1-ab)R$. *This shows that* $a \in \hat{\nabla}(R) \cap \hat{\Delta}(R)$.

(2) *Let* $a \in \hat{\nabla}(R)$. *Then* $(1-ab)R = R$ *for all* $b \in R$. *Suppose that* $a \notin \text{Tot}\,(R)$, *then there exists* $d \in R$ *such that* $0 \neq (ad)^2 = ad \in R$. *Thus,* $r(ad) = (1-ad)R = R$ *and* $ad = 0$ *a contradiction.*

Let $a \in \hat{\Delta}(R)$. *Then* $r(1-ba) = \{0\}$ *for all* $b \in R$.

If $a \notin \text{Tot}\,(R)$ *then there exists* $d \in R$ *such that* $0 \neq (da)^2 = da \in R$. *So* $da \in r(1-da) = \{0\}$ *a contradiction.*

(3) *by* (1) *and* (2).

Recall that a ring R *is a semipotent ring by Zhou [5], also called an* I_0-*ring by Nicholson [3], if every principal left (resp. right) ideal is not contained in* $J(R)$ *contains a nonzero idempotent.*

Lemma 3.4. *Let* R *be a semipotent ring. The following hold:*

(1) $J(R) = \hat{\nabla}(R)$.

(2) $J(R) = \hat{\Delta}(R)$.

(3) $\hat{\nabla}(R) = \hat{\Delta}(R)$.

Proof. (1). *It is clear that* $J(R) \subseteq \hat{\nabla}(R)$ *by Lemma 3.3. Let* $a \in \hat{\nabla}(R)$. *Then* $(1-ad)R = R$ *for all* $d \in R$. *If* $a \notin J(R)$ *then there exists* $b \in R$ *such that* $0 \neq b = bab$, *so* $0 \neq (ab)^2 = ab \in R$ *and* $r(ab) = (1-ab)R = R$. *So* $ab = 0$ *a contradiction. Thus,* $a \in J(R)$.

(2). *It is clear that* $J(R) \subseteq \hat{\Delta}(R)$ *by Lemma 3.3. Let* $a \in \hat{\Delta}(R)$. *Then* $r(1-da) = \{0\}$ *for all* $d \in R$. *If* $a \notin J(R)$ *then there exists* $b \in R$ *such that* $0 \neq b = bab$, *so* $0 \neq (ba)^2 = ba \in R$ *and that* $baR = r(1-ba) = \{0\}$. *So* $ba = 0$ *a contradiction. Thus,* $a \in J(R)$.

(3) *by* (1) *and* (2).

Corollary 3.5. *Let* R *be a semipotent ring and* $a \in R$. *The following hold:*

(1) $a \in J(R)$ *if and only if* $(1-ab)R = R$ *for all* $b \in R$, *if and only if* $r(1-ba) = \{0\}$ *for all* $b \in R$.

(2) $a \in J(R)$ *if and only if* $(1-ba)R = R$ *for all* $b \in R$, *if and only if* $r(1-ab) = \{0\}$ *for all* $b \in R$.

Proof *by Lemma 3.4.*

Theorem 3.6. (1). *For any ring* R *the following are equivalent:*

(i) $\hat{\nabla}(R) \subseteq J(R)$.

(ii) *For every* $a \in R$ *with* $1-a \in \hat{\nabla}(R)$; $r(a)=\{0\}$.

(2). *For any ring* R *the following are equivalent:*

(i) $\hat{\Delta}(R) \subseteq J(R)$.

(ii) *For every* $a \in R$ *with* $1-a \in \hat{\Delta}(R)$; $aR = R$.

Proof. (1) $(i) \Rightarrow (ii)$. *Let* $a \in R$, *with* $1-a \in \hat{\nabla}(R)$. *By assumption* $1-a \in J(R)$, *so*
$a = 1-(1-a) \in U(R)$. *Thus,* $r(a)=\{0\}$.
$(ii) \Rightarrow (i)$. *Let* $a \in \hat{\nabla}(R)$. *Then* $(1-ab)R = R$ *for all* $b \in R$. *Also, for all* $c \in R$; $(1-(ab)c)R = (1-a(bc))R = R$, *hence* $a \in \hat{\nabla}(R)$. *Thus,* $ab = 1-(1-ab) \in \hat{\nabla}(R)$ *by assumption* $r(1-ab)=\{0\}$, *so* $1-ab \in U(R)$ *for all* $b \in R$ *and that* $a \in J(R)$.
(2) $(i) \Rightarrow (ii)$. *Let* $a \in R$, *with* $1-a \in \hat{\Delta}(R)$. *By assumption* $1-a \in J(R)$, *so* $a = 1-(1-a) \in U(R)$. *Thus,* $aR = R$.
$(ii) \Rightarrow (i)$. *Let* $a \in \hat{\Delta}(R)$. *Then* $r(1-ab)=\{0\}$ *for all* $b \in R$. *Also, for all* $c \in R$;
$r(1-(ab)c) = r(1-a(bc)) = \{0\}$, *hence* $a \in \hat{\Delta}(R)$. *Thus,* $ab = 1-(1-ab) \in \hat{\Delta}(R)$ *by assumption* $(1-ab)R = R$, *so* $1-ab \in U(R)$ *for all* $b \in R$ *and that* $b \in J(R)$.

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