On Partially Pseudo Symmetric Contact Metric Three-Manifolds

Samir Abou Akl

Department of Basic Sciences- Faculty of Civil Eng-Damascus University-Syria

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ABSTRACT

In this paper we study some classes of partially pseudo symmetric contact three-manifolds .

In sections $3-1$ and $3-2$ we find necessary and sufficient conditions for a contact metric three-manifolds to be partially pseudo symmetric of the first and second type respectively . In section $3-5$ we find a necessary and sufficient conditions for a partially pseudo symmetric of the first type contact metric three-manifolds with constant ξ-sectional curvature to be pseudo symmetric. In section $3-6$ we find a sufficient condition for a contact metric three-manifolds M^3 with $\nabla_\xi \tau = 0$ to be pseudo symmetric.

Finally in section $3-7$ we find a necessary and sufficient condition for a contact metric 3-manifold with $Q\varphi = \varphi Q$ to be pseudo symmetric.

Key Words: Contact 3-Nabufikds Partially pseudo Symmetric.

دراسة حول المنطويات المترية التلامسية ثلاثية الأبعاد شبه المتناظرة جزئياً

سمير أبو عقل

قسم العلوم الأساسية – كلية الهندسة المدنية – جامعة دمشق - سورية

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الملخص

يتضمن هذا البحث دراسة بعض صنوف المنطويات المترية التلامسية ثلاثية الأبعاد شبه المتنـاظرة جزئياً. وجدتُ في البداية شروطاً لازمة وكافية حتى يكون منطو متري تلامسي ثلاثي ما شـبه متنـاظر جزئياً من النمط الأول أو الثاني. عرضت بعد ذلك ثلاثة صنوف من المنطويات المترية التلامسية ثلاثيـة الأبعاد:

آ – المنطويات المترية التلامسية شبه المتناظرة جزئياً من الـنمط الأول $(L = const.)$ **والتـي يكون تقوسها المقطعي** $-\ \xi$ **ثابتاً** $(\kappa = const. \neq L)$**. وقد وجدت أن الشرط اللازم والكافي حتـى تكون هذه المنطويات شبه متناظرة من النمط الثابت هو** $\kappa = 0$ **.**

ب – المنطويات المترية التلامسية التي تحقـق $\nabla_{\xi}\tau = 0$ **، وبينـت أنـه إذا حققـت الـشروط :**

$\lambda^2 - 1 - L = 0$ **و** ***L=const ≠0*** **(ومنه** $\lambda \neq 1$ **) و** $A = 0$ **أو** $B = 0$

فإن المنطويات تكون شبه متناظرة من النمط الثابت L **.**

جـ - المنطويات المترية التلامسية التي تحقق $Q\varphi = \varphi Q$ **، وقد وجـدت أن الـشرط الـلازم والكافي حتى تكون شبه متناظرة من النمط الثابت** $L \neq 0$ **هو** $(\lambda \neq 1)$ $\lambda^2 - 1 - L = 0$ **.**

الكلمات المفتاحية: المنطويات التلامسية الثلاثية شبه المتناظرة جزئياً.

1 – Introduction

A Riemannian manifold (M, g) is called pseudo symmetric if its curvature tensor R satisfies the condition [1]

(1) $\left(R(X,Y) \circ R\right)(U,V,W) = L[(X \wedge Y) \circ R)(U,V,W)]$

for all vector fields X, Y, U, V, W on M, where $L \in C^{\infty}(M)$ and

$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$

(2) $\left(R(X,Y) \circ R\right)(U,V,W) = R(X,Y)\left(R(U,V)\, W\right) -$

$- R\left(R(X,Y)U, V\right)W - R\left(U, R(X,Y)V\right)W - R(U,V)\left(R(X,Y)W\right)$

(3) $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$

(4) $\left((X \wedge Y) \circ R\right)(U,V,W) = (X \wedge Y)\left(R(U,V)\, W\right) -$

$- R\left((X \wedge Y)U, V\right)W - R\left(U, (X \wedge Y)V\right)W - R(U,V)\left((X \wedge Y)W\right)$

When L is constant, M is called a pseudo symmetric manifold of constant type .

If M is a contact metric manifold and (1) is satisfied only by certain special vector fields e.g.

(I) $Y = U = W = \xi$

(II) $Y = U = \xi$

(III) $Y = \xi$

then M is called partially pseudo symmetric contact metric manifold of the first type, of the second type or of the third type respectively .

Binh, T.Q; *et al.*, and others in [1] have studied some types of partially pseudo symmetric k-contact Riemannian manifolds .

In this paper we study some classes of partially pseudo symmetric contact three-manifolds .

2 – Preliminaries [5]

A contact metric manifold is a $(2n+1)$-dimensional differentiable manifold M^{2n+1} which carries a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ every where on M^{2n+1}. Every contact metric manifold has an underlying contact Riemannian structure (φ, ξ, η, g), where ξ

is a global vector field (called the characteristic vector field or Reeb vector field), φ a global tensor field of type $(1,1)$ and g a Riemannian metric (called associated metric) .

These structure tensors satisfy

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \qquad \eta(X) = g(\xi, X)$$
$$d\eta(X,Y) = g(X, \varphi Y), \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$
$$d\eta(\xi, X) = 0$$

Denoting by $\mathcal{L}$ the Lie derivation, we define the tensor field $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, which plays a fundamental role. h is symmetric , anticommuts with φ $(h\varphi + \varphi h = 0)$ and $h\xi = 0$. Also h vanishes if and only if ξ is killing. When ξ is killing ($\mathcal{L}_\xi g = 0$), the contact metric structure is said to be K-contact . If f is a real function and the almost complex structure J on $M^{2n+1} \times R$ defined by :

$$J(Y, f\frac{d}{dt}) = (\varphi Y - f\xi, \eta(Y)\frac{d}{dt})$$

is integrable, then the structure is said to be normal and the manifold is called Sasakian. A Sasakian manifold is a K-contact manifold . The converse is true only for dimension 3.

Let now $(M^3, \eta, g, \xi, \varphi)$ be a three-dimensional contact metric manifold. Let U be the open subset of M^3 where $h \neq 0$ and V the open subset of points $P \in M^3$ such that $h = 0$ in a neighborhood of P. For any point $m \in U \cup V$ there exists a local orthonormal basis $\{\xi, e, \varphi e\}$ of smooth eigen vectors of h in a neighborhood of m .On U we put :

$$he = \lambda e, \quad h\varphi e = -\lambda \varphi e$$

where λ is a non-vanishing smooth function which we suppose to be positive.

$2-1$ **PROPOSITION [3]**

On U we have :

$$\nabla_\xi \xi = 0 \quad , \quad \nabla_\xi e = -a\varphi e \quad , \quad \nabla_\xi \varphi e = a e$$

$$\nabla_e \xi = -(\lambda+1)e \ , \quad \nabla_e e = -\mu\varphi e \quad , \quad \nabla_e \varphi e = (\lambda+1)\xi + \mu e \tag{5}$$

$$\nabla_{\varphi e}\xi = -(\lambda-1)e \ , \quad \nabla_{\varphi e} e = (\lambda+1)\xi + \sigma\varphi e \quad , \quad \nabla_{\varphi e}\varphi e = -\sigma e$$

$$\nabla_\xi h = 2ah\varphi + \xi(\lambda)s$$

$$(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi + [\xi(A) + aB]e + [\xi(B) - aA]\varphi e$$

$$(\nabla_e Q)e = [e(A) + (\lambda+1)\xi(\lambda) + B\mu]\xi +$$

$$+ [e(\alpha + 2a\lambda) + 2\mu\xi(\lambda)]e +$$

$$+ [e\xi(\lambda) + 2a(\varphi e)(\lambda) + (2a - \lambda - 1)A]\varphi e \tag{6}$$

$$(\nabla_{\varphi e} Q)\varphi e = [(\varphi e)(B) + (\lambda-1)\xi(\lambda) + A\sigma]\xi +$$

$$+ [(\varphi e)\xi(\lambda) - 2a e(\lambda) + (1 - \lambda - 2a)B]e +$$

$$+ [(\varphi e)(\alpha - 2a\lambda) + 2\sigma\xi(\lambda)]\varphi e$$

$$R(\xi, e)\xi = -be + \xi(\lambda)\varphi e \quad R(e, \varphi e)e = B\xi + d\varphi e$$

$$R(\xi, \varphi e)\xi = \xi(\lambda)e - c\varphi e \quad R(\xi, e)\varphi e = -\xi(\lambda)\xi + Be$$

$$R(e, \varphi e)\xi = -Be + A\varphi e \quad R(\xi, \varphi e)\varphi e = c\xi - Ae \tag{7}$$

$$R(\xi, e)e = b\xi - B\varphi e \qquad R(e, \varphi e)\varphi e = -A\xi - de$$

$$R(\xi, \varphi e)e = -\xi(\lambda)\xi + A\varphi e$$

where a is a smooth function ,

$$\mu = -\frac{1}{2\lambda}[(\varphi e)(\lambda) + A] \quad , \qquad \sigma = -\frac{1}{2\lambda}[e(\lambda) + B]$$

$$A = \rho(\xi, e) = \mu(1 - \lambda - a) - \xi(\sigma) - \varphi e(a)$$

$$B = \rho(\xi, \varphi e) = \sigma(a - \lambda - 1) - \xi(\mu) + e(a)$$

$$\alpha = \frac{r}{2} - 1 + \lambda^2, \quad b = \lambda^2 - 1 - 2a\lambda$$

$$c = \lambda^2 - 1 + 2a\lambda, \quad d = \frac{r}{2} + 2\lambda^2 - 2$$

$r = 2[1-\lambda^2-\mu^2-\sigma^2-2a-e(\sigma)-(\varphi e)(\mu)]$

s is the $(1,1)$-tensor defined by :

$s\xi = 0$, $se = e$, $s\varphi e = -\varphi e$ and ρ , Q , r are the Ricci tensor , the corresponding Ricci operator and the scalar curvature respectively

$3-$ Partially pseudo symmetric contact metric 3-manifolds

$3-1$ THEOREM

Let M^3 be a contact metric three-manifold. Then M^3 is partially pseudo symmetric of the first type ($L = const.$) if and only if

(8) $A[(\lambda^2-1-2a\lambda)-L] = B\xi(\lambda)$

(9) $B[(\lambda^2-1+2a\lambda)-L] = A\xi(\lambda)$

<u>Proof</u>

Let

(10) $\big(R(X,\xi)\circ R\big)(\xi,Y,\xi) = L[\big((X\wedge\xi)\circ R\big)(\xi,Y,\xi)]$

for all vector fields X and Y on M^3 . Taking $X = e$ and $Y = \varphi e$ and using (2) , (3) , (4) and (7) we obtain :

$$\begin{aligned}
&\big(R(e,\xi)\circ R\big)(\xi,\varphi e,\xi) = R(e,\xi)\big(R(\xi,\varphi e)\xi\big)-\\
&-R\big(R(e,\xi)\xi,\varphi e\big)\xi - R\big(\xi,R(e,\xi)\varphi e\big)\xi - R(\xi,\varphi e)\big(R(e,\xi)\xi\big)\\
&= R(e,\xi)(\xi(\lambda)e - c\varphi e) - R(be-\xi(\lambda)\varphi e,\varphi e)\xi -\\
&-\big(R(\xi,\xi(\lambda))\xi - Be\big)\xi - R\big(\xi,\varphi e\big)(be-\xi(\lambda)\varphi e)\\
&= \xi(\lambda)(-b\xi + B\varphi e) - c(\xi(\lambda)\xi - Be) - b(-Be + A\varphi e)+\\
&+ B\big(-be+\xi(\lambda)\varphi e\big) - [b(-\xi(\lambda)\xi + A\varphi e) - \xi(\lambda)(c\xi - Ae)]\\
&= (cB-\xi(\lambda)A)e + 2(\xi(\lambda)B - bA)\varphi e
\end{aligned}$$

and

$$\begin{aligned}
&\big((e\wedge\xi)\circ R\big)(\xi,\varphi e,\xi) = (e\wedge\xi)\big(R(\xi,\varphi e)\xi\big)-\\
&-R\big((e\wedge\xi)\xi,\varphi e\big)\xi - R\big(\xi,(e\wedge\xi)\varphi e\big)\xi - R\big(\xi,\varphi e\big)\big((e\wedge\xi)\xi\big)\\
&= (e\wedge\xi)\big(\xi(\lambda)e - c\varphi e\big) - R(e,\varphi e)\xi - 0 - R(\xi,\varphi e)e\\
&= -\xi(\lambda)\xi - \big(-Be + A\varphi e\big) - \big(-\xi(\lambda)\xi + A\varphi e\big)\\
&= Be - 2A\varphi e
\end{aligned}$$

Making use of (10) we get :

$cB - A\xi(\lambda) = LB$, $bA - B\xi(\lambda) = LA$

and so , (8) and (9) hold .

Note that using (10) and taking $X = e$ and $Y = e$ we get the equation (9) taking $X = \varphi e$ and $Y = e$ we obtain (8) and (9) and taking $X = \varphi e$ and $Y = \varphi e$ we get (8). We conclude that all the possible choices of the vector field in the φ-basis $\{\xi, e, \varphi e\}$ give equations (8) and (9). So, if (8) and (9) hold, then (10) holds, that is M^3 is partially pseudo symmetric of the first type with $L = const$.

$3-2$ **REMARK .**

Note that taking

$X = X_1\xi + X_2 e + X_3\varphi e$ and $Y = Y_1\xi + Y_2 e + Y_3\varphi e$

and using the equations (2), (3), (4) and (7), the equation (10) becomes:

$$\{X_2Y_3(Bc - A\xi(\lambda)) + X_3Y_2 2(A\xi(\lambda) - Bc) + X_3Y_3(Ab - B\xi(\lambda))\}e +$$
$$+\{X_2Y_2(Bc - A\xi(\lambda)) + X_2Y_3 2(B\xi(\lambda) - Ab) + X_3Y_2(Ab - B\xi(\lambda))\}\varphi e =$$
$$= L\{(X_2Y_3B + X_3Y_2 2(-B) + X_3Y_3A)e +$$
$$+(X_2Y_2B + X_2Y_3 2(-A) + X_3Y_2A)\varphi e\}$$

From which we then get (8) and (9). Conversely if (8) and (9) hold then (10) holds for all vector fields X and Y, that is, M^3 is partially pseudo symmetric of the first type .

$3-3$ **THEOREM**

Let M^3 be a contact metric three-manifold. Then M^3 is partially pseudo symmetric of the second type where $L = const$. if and only if:

(11) $A[(\lambda^2 - 1 - 2a\lambda) - L] = B\xi(\lambda)$

(12) $B[(\lambda^2 - 1 + 2a\lambda) - L] = A\xi(\lambda)$

(13) $B^2 - [\xi(\lambda)]^2 + (\frac{r}{2} + 3\lambda^2 - 3 + 2a\lambda)(\lambda^2 - 1 - 2a\lambda - L) = 0$

(14) $A^2 - [\xi(\lambda)]^2 + (\frac{r}{2} + 3\lambda^2 - 3 - 2a\lambda)(\lambda^2 - 1 + 2a\lambda - L) = 0$

(15) $AB + \xi(\lambda)(\frac{r}{2} + 2\lambda^2 - 2 + L) = 0$

Proof

Let

(16) $\big(R(X,\xi)\circ R\big)(\xi,Y,Z) = L[\big((X\wedge\xi)\circ R\big)(\xi,Y,Z)]$

for all vector fields X , Y and Z on M^3 .

We apply (16) taking $X = e$, $Y = \varphi e$ and $Z = \xi$. We obtain (11) and (12) (theorem $3-1$). Taking now $X = e$, $Y = \varphi e$ and $Z = e$ and using (2), (3), (4) and (7) to calculate the left hand-side and the expression in the bracket on the right-hand side of (16) we obtain :

$$\begin{aligned}
&\big(R(e,\xi)\circ R\big)(\xi,\varphi e,e) = R(e,\xi)\big(R(\xi,\varphi e)\,e\big) - \\
&- R\big(R(e,\xi)\xi,\varphi e\big)e - R\big(\xi, R(e,\xi)\varphi e\big)e - R(\xi,\varphi e)\big(R(e,\xi)e\big) \\
&= R(e,\xi)(-\xi(\lambda)\xi + A\varphi e) - R(be - \xi(\lambda)\varphi e,\varphi e)e - \\
&- R\big(\xi,\xi(\lambda)\xi - Be\big)e - R(\xi,\varphi e)(-b\xi + B\varphi e) \\
&= -\xi(\lambda)\big(be - \xi(\lambda)\varphi e\big) + A(\xi(\lambda)\xi - Be) - b(B\xi + d\varphi e) + \\
&+ B(b\xi - B\varphi e) + b\big(\xi(\lambda)\,e - c\varphi e\big) - B(c\xi - Ae) \\
&= (A\xi(\lambda) - Bc)\xi + \{[\xi(\lambda)]^2 - B^2 - b(d+c)\}\varphi e
\end{aligned}$$

$$\begin{aligned}
&\big((e\wedge\xi)\circ R\big)(\xi,\varphi e,e) = (e\wedge\xi)\big(R(\xi,\varphi e)e\big) - R(e\wedge\xi)\xi,\varphi e)e - \\
&- R(\xi,(e\wedge\xi)\varphi e)e - R(\xi,\varphi e)((e\wedge\xi)e) \\
&= (e\wedge\xi)\big(-\xi(\lambda)\xi + A\varphi e\big) - R(e,\varphi e)e - \\
&- 0 - R(\xi,\varphi e)(-\xi) \\
&= -\xi(\lambda)e - (B\xi + d\varphi e) + \big(\xi(\lambda)e - c\varphi e\big) \\
&= -B\xi - (c+d)\varphi e
\end{aligned}$$

from which (13) follows .

In the same way, we use (16) taking $X = \varphi e$, $Y = e$ and $Z = \varphi e$. We obtain

$$-\big(Ab - B\xi(\lambda)\big)\xi - \{A^2 - [\xi(\lambda)]^2 + c(b+d)\}\varphi e = L[-A\xi - (b+d)e]$$

from which (14) follows instantly.

Finally, we put $X = e$, $Y = e$ and $Z = e$ in (15) and making use of (6) we get :

$$-\big(AB + d\xi(\lambda)\big)\varphi e = L\xi(\lambda)\varphi e$$

and so (15) holds .

Note that all the other possible choices of the φ-basis $\{\xi, e, \varphi e\}$, give again equation (11) - (15) .

So, if (11) - (15) hold, then (16) holds, that is, M^3 is partially pseudo symmetric of the second type where $L = const$.

$3-4$ **REMARK**

It is easy to verify that the conditions (11) - (15) are necessary and sufficient for a contact metric three-manifold M^3 to be pseudo symmetric .

$3-5$ **THEOREM**

Let M^3 be a partially pseudo symmetric of the first type ($L = const. \neq 0$) contact metric three-manifold with constant ξ-sectional curvature κ where $\kappa \neq L$ [5] .

Then M^3 is pseudo symmetric if and only if $\kappa = 0$.

<u>Proof</u>

Since M^3 is of constant ξ-sectional curvature κ, using (7), we have :

$$\kappa(e,\xi) = g\big(R(e,\xi)\xi, e\big) = \lambda^2 - 1 - 2a\lambda = \kappa$$

$$\kappa(\varphi e,\xi) = g\big(R(\varphi e,\xi)\xi, \varphi e\big) = \lambda^2 - 1 + 2a\lambda = \kappa$$

Adding and subtracting the two relations we obtain :

(17) $\lambda^2 - 1 = \kappa$, $a\lambda = 0$

Differentiating the first one with respect to ξ , e and φe , since $\lambda \neq 0$, we get :

(18) $\xi(\lambda) = e(\lambda) = \varphi e(\lambda) = 0$

The second relation of (17) yields $a = 0$. The relations (8) and (9) , using (17) and (18) become :

$A(\kappa - L) = 0$, $B(\kappa - L) = 0$

From which , since $\kappa \neq L$, it follows:

(19) $A = B = 0$

Therefore we get :

(20) $\mu = \sigma = 0$, $r = 2(1-\lambda^2)$

Finally , using (17) , (18) , (19) and (20) , we can conclude that (11) – (15) are satisfied if and only if $\kappa = \lambda^2 - 1 = 0$

$3-6$ THEOREM

Let M^3 be a contact metric three-manifold with $\nabla_\xi \tau = 0$ and satisfying the conditions :

(21) $\lambda^2 - 1 - L = 0$, *and* $A = 0$ *or* $B = 0$

where $L = const. \neq 0$ (and hence $\lambda \neq 1$) .Then M^3 is pseudo symmetric of constant type L .

Proof

Since M^3 satisfies $\nabla_\xi \tau = 0$ then [4]

(22) $a = 0$, $\xi(\lambda) = 0$

We assume $A = 0$ (if we assume $B = 0$,we proceed in the same way) . Differentiating the first relation of (21) with respect to e and φe , since $\lambda \neq 0$, we get :

(23) $e(\lambda) = \varphi e(\lambda) = 0$

We apply the well-known formula :

(24) $$\frac{1}{2} X(r) = \sum_{i=1}^{n} g((\nabla_{e_i} Q)e_i, X)$$

which holds for any vector field X of a n-dimensional Riemannian manifold, where $\{e_i\}$ is an arbitrary orthonormal basis, to compute $\frac{1}{2} e(r)$. Using (6) we obtain :

$$\frac{1}{2} e(r) = \xi(A) + aB + e(\frac{r}{2} - 1 + \lambda^2 + 2a\lambda) + 2\mu\xi(\lambda) +$$
$$+ (\varphi e)\big(\xi(\lambda)\big) - 2a\, e(\lambda) + (1 - \lambda - 2a)B$$

which because of (21), (22) and (23) takes the form $(1-\lambda)B = 0$. Therefore, since $\lambda \neq 1$, we get:

(25) $B = 0$

Finally, by virtue of (21), (22), (23) and (25) we can verify that (11)-(15) are satisfied, that is M^3 is pseudo symmetric of constant type L.

$3-7$ THEOREM

Let M^3 be a contact metric three-manifold with $Q\varphi = \varphi Q$. Then M^3 is pseudo symmetric of the constant type $L \neq 0$ if and only if:

$$\lambda^2 - 1 - L = 0 \qquad (\lambda \neq 1)$$

Proof

We know that the components of the Ricci operator Q, with respect to $\{\xi, e, \varphi e\}$, are given by :

$$Q\xi = 2(1-\lambda^2)\xi + Ae + B\varphi e$$

$$Qe = A\xi + (\frac{r}{2} - 1 + \lambda^2 + 2a\lambda)e + \xi(\lambda)\varphi e$$

$$Q\varphi e = B\xi + \xi(\lambda)e + (\frac{r}{2} - 1 + \lambda^2 - 2a\lambda)\varphi e$$

From which it follows easily that $Q\varphi - \varphi Q = 0$ if and only if :

(26) $A = B = a = \xi(\lambda) = 0$

We apply the formula (24) taking $X = e$ and $X = \varphi e$, using (26) we obtain : $e(\lambda) = (\varphi e)(\lambda) = 0$ and so

(27) $r = 2(1-\lambda^2)$

Finally, by virtue of (26) and (27) we can conclude that (11)-(15) are satisfied if and only if $\lambda^2 - 1 - L = 0$.

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