Non Resonant Case of Energy Exchange Between a Two–Level Atom and a Single Mode Radiation Field

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ABSTRACT

The quantum mechanical analysis for a non-resonant case of a single radiation mode interacting with a two-level atom in a cavity is studied. Expressions for the mean value of photon number and the time evolution of the atomic motion are given. The frequency of exchange energy between the atomic levels and the radiation field systems for non- resonant and resonant interactions is obtained. Results for all initial states of the atom-radiation field are presented. Results for resonant and non-resonant interactions are compared.

Key words: Two-level atom, Atomic motion, Non-resonant interaction**.**

حالة اللاتجاوب لتبادل الطاقة بين ذرة ثنائية السوية وحقل إشعاع وحيد النمط

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الملخص

درست مسألة تفاعل ذرة ثنائية السوية مع أنموذج حقل إشعاع وحيد النمط فـــي حالـــة اللاتجـــاوب ضمن ميكانيكا الكم. أعطيت عبارات القيمة المتوسطة للعدد الفوتوني والتطور الزمني للحركة الذرية. وتم الحصول على تواتر تبادل الطاقة بين السويات الذرية وحقل الأشعة في حـــالتي التجـــاوب واللاتجـــاوب. عرضت نتائج جميع شروط البدء للذرة - حقل الإشعاع. تمت المقارنة بين نتـــائج التفاعـــل اللاتجـــاوبي والتفاعل التجاوبي.

الكلمات المفتاحية: ذرة ثنائية السوية، الحركة الذرية، التفاعل اللاتجاوبي.

1. Introduction

It is clear that the problem of a single radiation mode interacting with a two-level atom in a cavity plays a fundamental role in the physics of laser [1]. Rabi has treated this problem for some time within the rotating-wave approximation [2]; he used a single classical radiation mode interacting with the atom. The quantum mechanical treatment within the same approximation of this system was performed by Jayens and Gummings [3], Scully and Lamb [4] treatment of the same model involved the evaluation of photon numbers representation. The quantum statistical properties of a single-atom maser [5] and a single-atom laser [6] were studied within the cavity. This system was treated as a spin-statistics via the combined operation of charge conjunction and time reversal [7]. The equation of motion of the spin vector of the two-level systems was treated in the presence of relaxation [8]. The resonant interaction of a two identical atoms with a single radiation field was solved via the coherent state method [9] . The full quantum mechanical treatment of the resonant interaction of a non identical two type of a two-level atom with a single radiation field was discussed [10] .

We consider in this work, the fully quantized model for a non - resonant interaction between the two level-atom and the radiation field, discuss the results for all possible initial states and compare them with the resonant interaction atom- radiation field.

The system consists of a single two-level atom resonant with the radiation field whose energy is given by $N \cdot \mathbf{h} w$ ($\mathbf{h}$ is the Planck's constant divided by $2p$) where N is the photons number, and w is given by:

$$w = \frac{E_2 - E_1}{\mathbf{h}} \qquad \qquad (1)$$

E_2 And E_1 are the exited $\left|\uparrow\right\rangle$ and ground $\left|\downarrow\right\rangle$ energy states of the atom respectively.

According to the atomic theory, the atom will be able to exchange a photon with the radiation field. However, suppose that the radiation field is not resonant with the atomic transition it is also possible for the atom to exchange a single photon whose energy is equal to the energy of the atomic transition. We describe the quantized radiation field by the vector $|N\rangle$, where the energy of the radiation field is given, by $N \cdot \hbar \Omega$. Here, we assume that $\Omega \neq \omega$, therefore, the approximation of the non-resonant interaction of atom-radiation field system, due to the inequality between Ω and ω, can be expressed as:

$$\Delta \omega = \Omega - \omega \qquad (1.1)$$

Schrödinger equation

The Hamiltonian of this system consists of the radiation field Hamiltonian in the absence of the atom $\hbar \Omega (a^+ a + 1/2)$, where a^+, a are the creation and annihilation operators [11] respectively, the Hamiltonian of free atom $\hbar w \, s^+ s$, where s^+, s are the spin ½ operators [12], and the atom-radiation field interaction Hamiltonian $\hbar g (a^+ + a)(s^+ + s)$, where g is the atom-radiation field coupling constant, which characterized [13] by the electric-dipole. Thus, the whole Hamiltonian of this system (neglecting the terms which do not conserve the energy [14] such $g \, a^+ s^+$ and $g \, a \, s$) can be written in the form:

$$H = \hbar \Omega \left(a^+ a + 1/2 \right) + \hbar w \, s^+ s + \hbar g \left(a^+ s + a \, s^+ \right) \qquad (2)$$

The wave function must satisfy the following Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} \left| y_1(t) \right\rangle = H \left| y_1(t) \right\rangle \qquad (3)$$

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substituting the transformation:

$$\left|y_1(t)\right\rangle = e^{-iH_1t}\left|f_1(t)\right\rangle \tag{3.1}$$

Where:

$$H_1 = \Omega\left(a^+a + \frac{1}{2}\right) + w\, s^+s + \Delta w\left(s^+s - \frac{1}{2}\right) \tag{3.2}$$

Into Schrodinger equation (3), and using the Hamiltonian (3.2), then we will obtain the following Schrodinger equation for the new wave function $\left|f_1(t)\right\rangle$:

$$i\frac{\partial}{\partial t}\left|f_1(t)\right\rangle = e^{iH_1t}H_2 e^{-iH_1t}\left|f_1(t)\right\rangle \tag{3.3}$$

Where

$$H_2 = g\left(a^+s + a\, s^+\right) - \Delta w\left(s^+s - \frac{1}{2}\right) \tag{3.4}$$

We can see that the system's Hamiltonian unchanged, we mean that, $H = \mathbf{h}\left(H_1 + H_2\right)$, in addition, H_1 and H_2 obey the commutation relation, $\left[H_1, H_2\right] = 0$, using this commutator in the following relation [15]:

$$e^{iH_1t}H_2 e^{-iH_1t} = H_2 + \frac{(it)}{1}\left[H_1, H_2\right] + \frac{(it)^2}{1\times 2}\left[H_1,\left[H_1, H_2\right]\right] + \mathbf{L} \tag{3.5}$$

putting this result in the Schrodinger equation (3.3), which can be rewritten in the following suitable form:

$$i\frac{\partial}{\partial t}\left|f_1(t)\right\rangle = H_2\left|f_1(t)\right\rangle \tag{4}$$

2. The wave function and the Equation of motion:

We assume that the interaction is (turned on) at $t = 0$, and the system has the atom in the ground state energy with N photons (this situation can be represented by $|\downarrow, N\rangle$). At later time, it is possible for the atom to rise to the excited energy state, with simultaneous absorption of a single photon; (this situation can be represented by $|\uparrow, N-1\rangle$). Thus, the atom – radiation field system oscillates between these two states, and the wave function $|f_1(t)\rangle$ is just a linear summation of these states:

$$\left| f_1(t) \right\rangle = x_1(t) \left| \downarrow, N \right\rangle + y_1(t) \left| \uparrow, N-1 \right\rangle \tag{5}$$

$x_1(t)$ And $y_1(t)$ are the amplitude of the states $|\downarrow, N\rangle$ and $|\uparrow, N-1\rangle$ respectively. If we substitute the wave function (5) into Schrodinger equation (4), then we will obtain the equation of motion for the amplitudes $x_1(t)$ and $y_1(t)$:

$$i \frac{\partial}{\partial t} \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \tag{5.1}$$

Where

$$a = \frac{\Delta w}{2}$$

$$b = g\sqrt{N} \tag{5.1.1}$$

According to the wave function (5) the initial condition becomes $\left| f_1(t) \right\rangle = |\downarrow, N\rangle$, which expressed in the amplitudes representation $x_1(0) = 1$ and $y_1(0) = 0$, therefore, the solution can be written as:

$$x_1(t) = Cos\,(l\ t) - i\frac{\Delta w}{2\,l}\,Sin(l\ t)$$

$$\tag{6}$$

$$y_1(t) = -i\,\frac{g\sqrt{N}}{l}\,Sin\,(l\ t)$$

Where

$$l = \sqrt{Ng^2 + \frac{\Delta w^2}{4}}$$

$$\tag{6.1}$$

Notice that the wave function $\left|f_1(t)\right\rangle$ is normalized:

$$\left\langle j_1(t)\,\middle|\,j_1(t)\right\rangle = \left|x_1^2(t)\right| + \left|y_1^2(t)\right| = 1 \tag{7}$$

Here we used the orthogonal property of two states via the relations:

$$\left\langle s,N\,\middle|\,N',s'\right\rangle = d_{N,N'}\ d_{s,s'} \quad \& \quad s = \uparrow,\downarrow\ ;\ s' = \uparrow,\downarrow \tag{7.1}$$

If we use the transformation (3.1) we can see that the wave function $\left|f_1(t)\right\rangle$ is an Eigenfunction of H_1, therefore, if we operate on the wave function $\left|f_1(t)\right\rangle$ with the Hamiltonian H_1 we will obtain the wave function $\left|f_1(t)\right\rangle$ (eigenfunction) with the eigenvalue $\Omega\cdot\left(N+\frac{1}{2}\right) - \frac{\Delta w}{2}$, so the wave function $\left|y_1(t)\right\rangle$ is given by:

$$\left|y_1(t)\right\rangle = e^{-i\left[\Omega\left(N+\frac{1}{2}\right) - \frac{\Delta w}{2}\right]t}\,\left|f_1(t)\right\rangle \tag{7.2}$$

Thus, the wave function $\left|y_1(t)\right\rangle$ is normalized too:

$$\left\langle y_1(t)\,\middle|\,y_1(t)\right\rangle = 1 \tag{7.3}$$

In order, if we use the relations (7.1), (7.2) and (7.3), then we can write the energy of the radiation field in form:

$$\hbar\Omega \left\langle y_1(t) \mid \left(a^+a + 1/2\right) \mid y_1(t)\right\rangle = \hbar\Omega \left[N + 1/2 - \frac{N g^2}{l^2} Sin^2(l\ t)\right] \qquad (8)$$

Similarly, the energy of the atom can be written as:

$$\hbar w \left\langle y_1(t) \mid s^+s \mid y_1(t)\right\rangle = \hbar w \left[\frac{N g^2}{l^2} Sin^2(l\ t)\right] \qquad (9)$$

Let us consider the situation where the interaction is (turned on) at $t = 0$, and the system has the atom in the excited energy state with $N - 1$ photons (this state represented by $|\uparrow, N-1\rangle$), then the atom will emit a single photon with simultaneous transition to the ground energy state $|\downarrow, N\rangle$. Thus the new wave function can be written as:

$$\left| f_2(t)\right\rangle = x_2(t) |\uparrow, N-1\rangle + y_2(t) |\downarrow, N\rangle \qquad (10)$$

Where we used the following Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} \left| y_2(t)\right\rangle = H \left| y_2(t)\right\rangle \qquad (11)$$

And the transformation:

$$\left| y_2(t)\right\rangle = e^{-iH_1 t} \left| f_2(t)\right\rangle \qquad (11.1)$$

If we substitute (11.1) into (11) and using (10), then we will obtain the following equations of motion for the wave function $\left| y_2(t)\right\rangle$:

$$i \frac{\partial}{\partial t} \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -a & b \\ b & a \end{pmatrix} \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \qquad (12)$$

And the solution of the differential equations (12), under the following initial conditions $x_2(0) = 1$ and $y_2(0) = 0$, defined by:

$$x_2(t) = Cos\ (l\ t) + i\frac{\Delta w}{2l}\ \ Sin\ (l\ t)$$

(13)

$$y_2(t) = -i\frac{g\sqrt{N}}{l}\ Sin\ (l\ t)$$

Note that l, H_1 and H_2 are the same.

We followed the same foot steps and found the energy of the radiation field to this situation in the form:

$$\hbar\Omega\langle y_2(t)\,|\,(a^+a+1/2)\,|\,y_2(t)\rangle = \hbar\Omega\left[N - 1/2 + \frac{Ng^2}{l^2}Sin^2(l\ t)\right]$$

(14)

And the energy of the atom as:

$$\hbar w\langle y_2(t)\,|\,s^+s\,|\,y_2(t)\rangle = \hbar w\left[1 - \frac{Ng^2}{l^2}Sin^2(l\ t)\right]$$

(15)

3. Resonant case:

In the resonant case ($\Delta w = 0$), we rewrite the Hamiltonian H_1 and H_2 in form

$$H_1 = w\left(s^+s + a^+a + \frac{1}{2}\right)$$

(16)

$$H_2 = g\left(a^+s + a\ s^+\right)$$

(17)

Thus, we rewrite the solutions of the wave function $|y_1(t)\rangle$ of the resonant case in form:

$$\left| y_1(t) \right\rangle = e^{-iw\left(N+\frac{1}{2}\right)t} \left| f_1(t) \right\rangle \tag{18}$$

Where $\left| f_1(t) \right\rangle$ has the same form as (5) with the following amplitudes [12,13]:

$$x_1(t) = Cos\,(l_r\,t) \tag{18.1}$$

$$y_1(t) = -i\,Sin\,(l_r\,t)$$

Where

$$l_r = g\sqrt{N} \tag{18.2}$$

And the energy of the radiation field becomes:

$$\mathbf{h}\Omega \left\langle y_1(t) \mid \left(a^+ a + 1/2\right) \middle| y_1(t) \right\rangle = \mathbf{h}w \left[N + 1/2 - Sin^2(l_r\,t) \right] \tag{18.3}$$

Also, the energy of the atom will be:

$$\mathbf{h}w \left\langle y_1(t) \mid s^+ s \middle| y_1(t) \right\rangle = \mathbf{h}w \left[Sin^2(l_r\,t) \right] \tag{18.4}$$

Similarly, we rewrite the wave function $\left| y_2(t) \right\rangle$ of the resonant case in form:

$$\left| y_2(t) \right\rangle = e^{-iw\left(N+\frac{1}{2}\right)t} \left| f_2(t) \right\rangle \tag{19}$$

Where $\left| f_2(t) \right\rangle$ has the same form as (10) with the following amplitudes:

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$$x_2(t) = Cos \ (l_r \ t)$$

$$(19.1)$$

$$y_2(t) = -i \ Sin \ (l_r \ t)$$

The energy of the radiation field becomes:

$$\mathbf{h}\Omega \langle y_2(t) \mid \left(a^+ a + 1/2 \right) \mid y_2(t) \ \rangle = \mathbf{h}w \ \left[N - 1/2 + Sin^2(l_r \ t) \right] \qquad (19.2)$$

And the energy of the atom:

$$\mathbf{h}w \langle y_2(t) \mid s^+ s \mid y_2(t) \ \rangle = \mathbf{h}w \left[\ Cos^2(l_r \ t) \right] \qquad (19.3)$$

4. Discussion:

Equations (8), (9) show that, the energy oscillates in non-resonant case, with frequency (Rabi frequency [2,3]) $l \ / \ 2$ between the atom and the radiation field. And equation (18.3) and (19.2) show that, the energy oscillates with frequency $l_r \big/ 2$ in resonant case. Equations (6.1), (18.2) show that, l and l_r related to one another via the relation:

$$l = l_r \left(1 + \frac{\Delta w^2}{4 N g^2} \right)^{1/2} \qquad (20)$$

Note that, if Δw very small then l and l_r will be very close. Fig.1 shows plot of l as a function of Δw, the axis Δw represents the plot of l_r, where we know that, if $\Delta w \to 0$ then $l \to l_r$, and if Δw increased l_r remain constant.
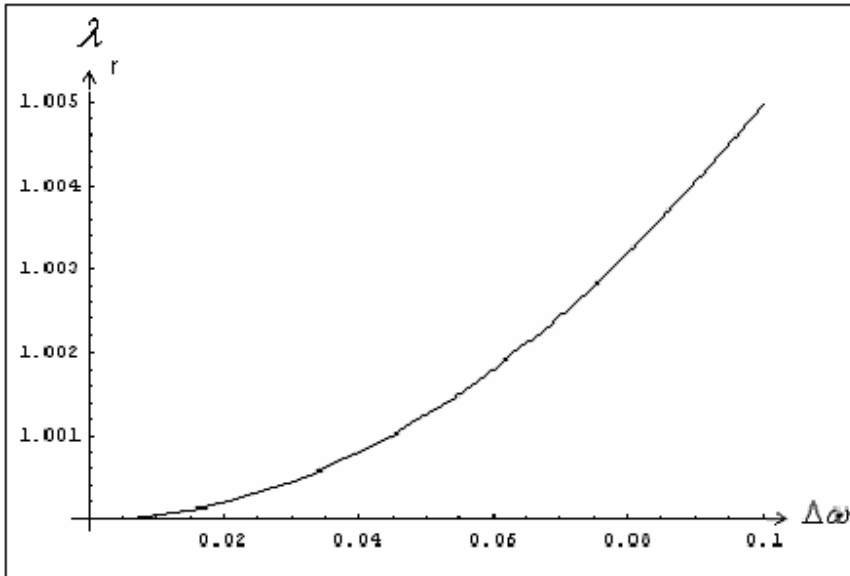
Fig.1. plot of l **as function of** ΔW, $g = 0.5, N = 1$.

Both fig.2 and fig.3 show diagram of the time development of the probability density $|x_1(t)|^2$ and $|y_1(t)|^2$ of the states $|\downarrow, N\rangle$ and $|\uparrow, N-1\rangle$ (respectively) of the wave function $|y_1(t)\rangle$.

We can see that, a delay in the exchange of the energy between the atom and the radiation field in non-resonant interaction due to Δw, as when $\Delta w = 0.08$ then the system of non-resonant interaction exchanges energy slower than the same picture when $\Delta w = 0.04$, which in turn exchanges energy slower than the same picture of the resonant case ($\Delta w = 0$). Thus, the delay becomes more obvious and slower as Δw increases.

We note that the same picture to the non-resonant interaction of the wave function $|y_2(t)\rangle$, where the diagram of $|x_2(t)|^2$ is the same

diagram as $\left|x_1(t)\right|^2$, and the diagram of $\left|y_2(t)\right|^2$ is the same diagram as $\left|y_1(t)\right|^2$. The only difference is that, $\left|x_2(t)\right|^2$ is the probability density of the state $\left|\uparrow, N-1\right\rangle$ and $\left|y_2(t)\right|^2$ is the probability density of the state $\left|\downarrow, N\right\rangle$, with initial condition $\left|y_2(0)\right\rangle = \left|\uparrow, N-1\right\rangle$ or $x_2(0) = 1$ and $y_2(0) = 0$.

Note that figure (2) shows that, for $\Delta w = 0.08$, then, according to equation (20) $1 \approx 1.003\, 1_r$, at $t \approx 970.772$, where, the probability density of resonant interaction is $\left|x(t)\right|^2 \approx 0.00009$, and the probability density of non-resonant interaction is $\left|x(t)\right|^2 \approx 0.999$, thus, the change of the probability density of non-resonant interaction inversed relative to the signal of the probability density of resonant interaction.

Similarly, figure (3) shows that for $\Delta w = 0.04$, where, $1 \approx 1.0008\, 1_r$, the inversed point becomes at $t \approx 4036.84$, in this case, the probability density of resonant interaction is $\left|x(t)\right|^2 \approx 0.0019$ and the probability density of non-resonant interaction is $\left|x(t)\right|^2 \approx 0.9999$. So, as $\Delta w \to 0$, then, $1 \to 1_r$, also, figure (1) shows that 1 very close to 1_r for $\Delta w \leq 0.008$ where $1 \leq 1.00003\, 1_r$.

5. Conclusion

So the approximation dependes on the value of Δw, when Δw tends to zero then 1 tends to 1_r and the inversed point tends to the infinity, thereby, the non-resonant signal close to the resonant signal, however, the Green function calculation, for the linear polarizability of a two-level atom interact with light [16] has been

used a good approximation for frequency close to the resonance, such $\Delta w = 0$, it has been shown here that, this approximation is not close to the resonant interaction, it represents the resonant interaction picture.
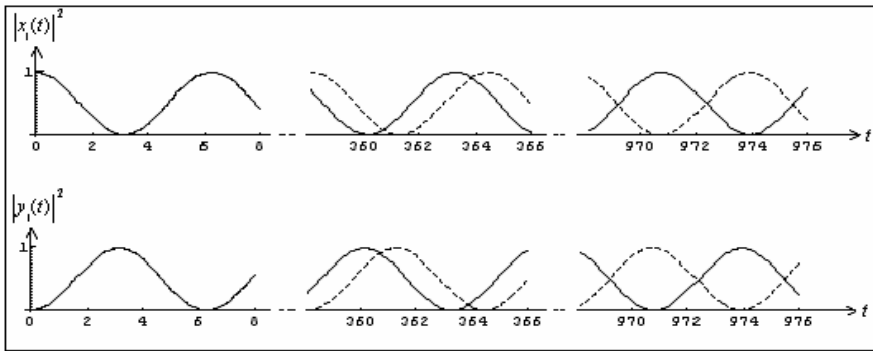

Fig 2. probability density $\left|x_1(t)\right|^2$ and $\left|y_1(t)\right|^2$ of the wave function with the initial state $\left|y_1(0)\right\rangle = \left|\downarrow, N\right\rangle$, where, $g = 0.5, N = 1$. Sold line $\Delta w = 0.08$, and dot line $\Delta w = 0$ (resonant case).
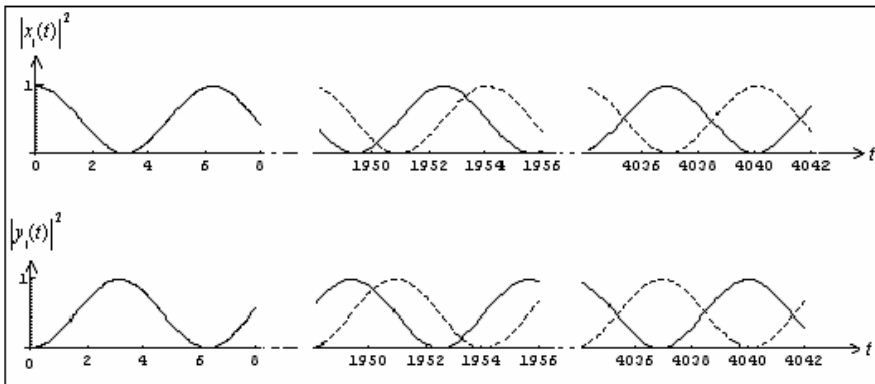

Fig 3. probability density $\left|x_1(t)\right|^2$ and $\left|y_1(t)\right|^2$ of the wave function with the initial state $\left|y_1(0)\right\rangle = \left|\downarrow, N\right\rangle$, where, $g = 0.5, N = 1$. Sold line $\Delta w = 0.04$, and dot line $\Delta w = 0$ (resonant case).

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