Endomorphism Rings Of Modules

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ABSTRACT

The objective of this paper is to continue our study for a right I_1-rings and to generalize the concept of I_1-rings to modules. We call a ring R a right I_1- ring if every right annihilator for any element of R contains a nonzero idempotent.

We call a nonzero projective module P an I_1-module, if every nonzero submodule of P contains a nonzero direct summand of P. Our main result is that any projective module over a right I_1-ring is an I_1-module, with zero radical. On the other hand, we prove that, a ring R is a right I_1-ring if and only if endomorphism ring of any projective (or equivalently, any free) $R-$module is a right I_1-ring.

Finally, we prove that, any free module F with zero radical is an I_1-module if and only if endomorphism ring of F is a right I_1-ring.

Key Words: *Right I_1-ring, Radical of module, Projective module, Free module, Endomorphism ring of module, Annihilator.*

حلقة التشاكلات للمودولات

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الملخص

موضوع هذا البحث هو متابعة دراسة الــ I_1-حلقات من اليمــين وتعمــيم هــذا المفهــوم علــى المودولات. نقول عن حلقة ما: إنها I_1-حلقة من اليمين إذا حوى كل عادم يمينــي لأي عنــصر منهــا عنصراً جامداً مغايراً للصفر. وقد عُمّم هذا المفهوم على المودولات الإسقاطية. سوف نقول عن المودول الإسقاطي المغاير للصفر إنه I_1-مودول إذا كان كل مودول جزئي منه ومغاير للصفر يحوي حداً مباشراً مغايراً لصفر هذا المودول. وقد أُثبت أن كل مودول إسقاطي فوق I_1-حلقة من اليمين هو I_1-مودول، فضلاً عن ذلك، أُثبت أن كل I_1-مودول إسقاطي أساسه معدوم. وهذا يبين لنا أيضاً أن أساس جاكبسون للــ I_1-حلقات من اليمين يساوي الصفر. كما أنه أُثبت أن الشرط اللازم والكافي كي تكون حلقــة مــا هي I_1-حلقة من اليمين هو أن تكون حلقة التشاكلات لأي مودول إسقاطي (حر) فوقها هي I_1-حلقة من اليمين، والذي يكافئ بدوره أن حلقة التشاكلات لأي مودول إسقاطي (حر) هي حلقة بيير المعممة من اليمين. وفي النهاية أُعطي الشرط اللازم والكافي كي يكــون المــودول الحــر ذو الأســاس المعــدوم، I_1-مودولاً، وذلك من خلال الصورة المباشرة، والنواة لأي تشاكل لهذا المودول وأيضاً من خلال حلقة التشاكلات لهذا المودول.

الكلمات المفتاحية: I_1-حلقة من اليمين، أساس المــودول، المــودول الإســقاطي، المودول الحر، حلقة التشاكلات لمودول، عادم عنصر.

Introduction

Throughout this paper, unless otherwise indicated, all modules over a ring R will be understood to be right $R-$modules. R will always associative have unit, and every module will be unitary. If M is an R- module, the radical of M, denoted $J(M)$, is defined to be intersection of all maximal submodules of M. It may happen that M has no maximal submodules in which case J(M)=M. Thus, for a ring R, J(R) is the Jacobson radical of R.

$R-$module P is called projective, if for any ipemmorphism of $R-$modules $f:B\rightarrow C$ and any homomorphism $g:P\rightarrow C$, there exists a homomorphism $R-$modules $h:P\rightarrow B$ such that $g = fh$. If P is a projective $R-$module then P is a direct summand of a free module [1, Theorem 2.2] and hence $J(P)=PJ(R)$. Bass proved [2, proposition 2.7] that if $P\neq o$ is a projective module then $PJ(R)\neq P$. Thus every projective module has a maximal submodule.

For any element $a\in R$, we denoted the right annihilator of a in R by $r(a)=\{x : x\in R;\ ax=o\}$. Similarly, the left annihilator of a in R is denoted by $l(a)=\{y : y\in R;\ ya=o\}$.

A ring R is called a generalized right Baer ring[4], if any right annihilator contains a non-zero idempotent.

In this paper we will continue our study of I_1-rings [3], through attempt to generalize the concept of I_1- rings to modules in order to get a complete structure of a right I_1-rings.

Lemma 1. *[3, lemma 1]. For any ring R, the following conditions are equivalent:*

1. *For any element $a\in R$, there exists in R an idempotent $f\neq 1$ such that $l(a)\subseteq Rf$.*
2. *For any element $b\in R$, there exists in R an idempotent $e\neq o$ such that $e\in r(b)$.*
3. *For any element $d\in R$, there exists in R an idempotent $g\neq 1$ such that $d=dg$.*

Definition 1. *A ring R is called a right I_1-ring [3], if it satisfies the equivalent conditions of lemma 1.*

Lemma 2. *If R is a right I_1-ring, then $J(R)=o$.*

Proof. Assume that $J(R)\neq o$, then there exists $a\in J(R), a\neq o$. Since R is a right I_1-ring and $(1-a)\in R$ then by lemma 1, there exists in R an idempotent $g\neq 1$ such that $(1-a)=(1-a)g$, therefore $(1-g)=a(1-g)$. Since $g\neq 1$ is an idempotent, then $(1-g)\neq o$ is an idempotent such that $(1-g)=a(1-g)\in \ aR\subseteq J(R)$, this is contradicting that J(R), which does not contain nonzero idempotents. Hence $J(R)=o$. Thus our proof is completed.

Theorem 1. *Let P be a projective R- module. If R is a right I_1-ring, then every submodule of P contains a direct summand of P.*

Proof. Since R is a right I_1-ring, then by lemma 2, $J(R)=o$, therefore $J(P)=PJ(R)=o$.

First we prove that any submodule of finitely generated free $R-$module contains a direct summand.

Let F be a free R- module with basis $\{x_1,x_2,\ldots,x_n\}$, then $F=x_1R\oplus\ldots\oplus x_nR$.

Let $K\neq o$ be a submodule of F, then there exists $a\in K$, and $a\neq o$.

We put $a=x_1r_1+x_2r_2+\ \ldots+x_nr_n$, $r_i\in R,(i=1,2,\ldots n)$. Without loss of generality, we may assume that $r_1\neq o$. Since $1-r_1\in R$ and R is a right I_1- ring, there exists an idempotent $1\neq e\in R$ such that $1-r_1=r_1(1-e)$, by lemma 1, therefore $1-e=r_1(1-e)$ and $1-e$ is a nonzero idempotent of R. We put $g=1-e$, then $g=r_1g\in r_1R$, and $ag=x_1g+x_2r_2g+\ldots+x_nr_ng$.

Since 1 = g+ (1 - g) we can easily see that $F = agR \oplus x_1(1-g)R \oplus x_2R \dots \oplus x_nR$ *. Thus* agR *is a direct summand of* F *and* $agR \subseteq aR \subseteq K$ *.*

Second we prove that any submodule of a free $R-$*module contains a direct summand. Let* G_R *be a free module with basis* $\{x_l\}_{l\in\Lambda}$ *and* $A \neq o$ *be a submodule of* G *, then there exists* $a \in A$*, and* $a \neq o$*. We put* $a = x_{i_1}r_1 + x_{i_2}r_2 + \dots + x_{i_n}r_n$, $r_i \in R$, $(i = 1, 2, \dots, n)$ *and* $G_n = x_{i_1}R \oplus x_{i_2}R \oplus \dots \oplus x_{i_n}R$ *. Since* $aR \subseteq G_n$*, there exists a submodule* N *of* aR *such that it is a direct summand of* G_n *by first case. Also since* G_n *is a direct summand of* G*,* N *is a direct summand of* G *. Finally, we shall complete the proof of this theorem. Let* P *be a projective module, then* P *is a direct summand of a free module F. We put* $F = P \oplus F_o$ *, where* F_o *is a submodule of F. Let* M *be a submodule of P, then* M *is a submodule of* F *, by second case there exists a direct summand* Q *of* F *such that* $Q \subseteq M$ *. We put* $F = Q \oplus Q'$*. Then* $P = Q \oplus (P \cap Q')$ *by modular law, that is,* Q *is a direct summand of* P *. Hence any submodule of* P *contains a direct summand of* P *.*

Definition 2. *We call a projective* $R-$*module* P *,* I_1-*module if any nonzero submodule of* P *contains a nonzero direct summand of* P*.*

From theorem 1 the following can be obtained

Corollary 1. *Any projective module over a right* I_1-*ring, is an* I_1-*module.*

Proposition 1. *Let* $\{P_i\}_{i\in I}$ *be a family of projective R- modules. Then* $P = \sum_{i\in I} \oplus P_i$ *is an* I_1-*module if and only if for any* $i \in I$ *,* P_i *is an* I_1-*module.*

Proof. *Suppose P an I_1-module. Let A be a submodule of P_j where $j \in I$, then A is a submodule of P. Since P is an I_1-module, then A contains a direct summand N of P, therefore $P = N \oplus M$ where M is a submodule of P. Since $N \subseteq A \subseteq P_j$ then $P_j = N \oplus (M \cap P_j)$. This implies that N is a direct summand of P_j and $N \subseteq A$.*

Suppose conversely P_i is an I_1-module for each $i \in I$. Let $K \neq o$ be a submodule of P, then there exists $a \in A$ and $a \neq o$, $a = a_{i_1} + a_{i_2} + \ldots + a_{i_t}$, where $a_{i_j} \in P_{i_j}$, ($j = 1,2, \ldots t; i_j \in I$). Without loss of generality, we may assume $a_{i_1} \neq o$. Since P_{i_1} is an I_1-module, then $a_{i_1}R$ contains a direct summand of P_{i_1} which is a direct summand of P. Thus our proof is completed.

The next result gives a complement structure of a right I_1-rings.

Theorem 2. *For any ring R the following conditions are equivalent:*

1 – R is a right I_1-ring.

2 – $End_R(F)$ is a right I_1-ring for any free R- module F.

3 – $End_R(P)$ is a right I_1-ring for any projective R- module P.

4– $End_R(F)$ is a generalized right Bear ring for any free R- module F.

5 –$End_R(P)$ is a generalized right Bear ring for any projective $R-$module P.

Proof. $(2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$ *are given in [3, Theorem 8].*

(1)$\Rightarrow$(2). Let F be a free $R-$module, since F is a projective R-module and R is a right I_1-ring, then by corollary 1, any submodule of F contains a direct summand of F, and also $Ker f$ contains a

direct summand of F *, for any* $f \in End_R(F)$*, by [3, Theorem 8],* $End_R(F)$ *is a right* I_1-*ring.*

(2)$\Rightarrow$*(1). Follows from the fact* $R \cong Hom_R(R,R)$ *and* R_R *is a free* $R-$ *module.*

Definition 3. *Tow rings* R_1, R_2 *are called Morita equivalently if there exists a finitely generated projective module* R_1-*module* P *such that* $R_2 \cong End_{R_1}(P)$.

Proposition 2. *Let* R_1 *and* R_2 *are rings with identities and* R_1 *a right* I_1-*ring. If* R_2 *is Morita equivalent to* R_1*, then* R_2 *is a right* I_1-*ring.*

Proof. Since R_2 *is Morita equivalent to* R_1*, there exists a finitely generated projective module* P *as a right* R_1-*module such that* $R_2 \cong End_{R_1}(P)$*. Also since* R_1 *is a right* I_1-*ring, then by Theorem 2,* $End_{R_1}(P)$ *is a right* I_1-*ring, so* R_2 *is a right* I_1-*ring.*

Corollary 2. *If* R *is a right* I_1-*ring, then the ring of all* $n \times n$ *matrices* $M_n(R)$ *is a right* I_1-*ring.*

Proof. Let R^n *be the direct sum of* n *copies of* R *.Since* R^n *is a finitely generated projective right* $R-$*module, follows from corollary 1, if* R *is a right* I_1-*ring, then* $End_R(R^n)$ *is a right* I_1-*ring. Therefore* $M_n(R)$ *is a right* I_1- *ring, since* $M_n(R) \cong End_R(R^n)$.

Corollary 3. *If* P *is a nonzero projective* I_1-*module, then* $J(P)=o$.

Proof. Let P *be a nonzero projective* I_1-*module. Suppose that* $J(P) \neq o$ *, then there exists,* $x \in J(R)$ *and* $x \neq o$ *therefore* xR *is a*

nonzero submodule of P *, by definition,* xR *contains a nonzero direct summand A of P which is projective. Since* $A \subseteq xR \subseteq J(P)$ *, and A is a direct summand of P, then* $J(A) = A \cap J(P)$
$= A$*. This is a contradiction . Thus* $J(P) = o$.

Theorem 3. *Let F be a free R- module with* $J(F) = o$ *, and* $S = End_R(F)$ *. The following conditions are equivalent:*

1 – F *is an* $I_1 -$*module.*

2 – For any $f \in S$ *,* $Ker f$ *contains a direct summand of F.*

3 – For any $f \in S$*,* $Im f$ *contains a direct summand of F.*

4 – $S = End_R(F)$ *is a right* $I_1 -$*ring.*

Proof. (1)$\Rightarrow$*(2). Let* $f \in S$ *, since* $Kerf$ *is a submodule of F, then* $Kerf$ *contains a direct summand of F.*

(2)$\Rightarrow$*(3). Let fÎS, then* $(1-f) \in S$ *, therefore* $Ker(1-f)$ *contains a direct summand of F. Since* $Ker(1-f) \subseteq Im f$ *,then* $Im f$ *contains a direct summand of F.*

(3)$\Rightarrow$*(1). Let* $A \neq o$ *be a submodule of F, since* $J(F) = o$ *, then there exists a maximal submodule M of F such that* $A \not\subset M$ *. Thus* $F = A + M$ *by [4, lemma 2.2],* $S = \hat{A} + \hat{M}$ *where* $\hat{A} = Hom_R(F, A)$ *,* $\hat{M} = Hom_R(F, M)$ *. Since* $1 \in S$ *, then* $1 = j + y$ *for some* $j \in \hat{A}$*,* $y \in \hat{M}$ *. Therefore,* $Im j$ *contains a direct summand of F. Since* $Im j \subseteq A$ *, A contains a direct summand of F. This shows that F is an* $I_1 -$*module.*

(2)$\Leftrightarrow$ *(4). By [3 , Proposition 5].*

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