On Contact Metric Three-Manifolds Satisfying Certain Curvature Conditions

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ABSTRACT

In this article we study the contact metric three-manifolds M^3 **satisfying certain curvature conditions especially** $x(l) = const.$ **In section** $3-2$ **we find four conditions such that every one is necessary and sufficient for a contact metric three-manifold satisfying**

$$(*) \qquad x(l) = A = B = 0, \quad l^2 - 1 + 2al = 0$$

to be semi-symmetric. In section $3-4$ **we prove that if a contact metric three-manifold** M^3 **satisfies** $(*)$ **then either** $a = 0$ **and** M^3 **is semi-symmetric or** $a \neq 0$ **and in this case *a* is constant and** M^3 **cannot be semi-symmetric . In section** $3-5$ **we prove that if a semi-symmetric contact metric three-manifold satisfies**

$$x(l) = const. \qquad A = const. \qquad B = const.$$

(in section 3-6 if $x(l) = const.\ e(l) = const.\ j\ e(l) \neq 0.$ **)**

then $a = x(l) = A = B = 0$

Key Words: Contact 3-Manifolds and Curvature Conditions.

دراسة حول المنطويات المترية التلامسية ثلاثية الأبعاد التي يحقق تقوسها بعض الشروط الخاصة

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الملخص

ندرس في هذا البحث، المنطوي المتري التلامسي ثلاثي الأبعاد M^3 الذي يحقق تنـــسور تقوســـه بعض الشروط ولاسيما $x(l) = const.$.

نورد بداية أربعة شروط ، كل منها لازم وكافٍ حتى يكون منطو متري تلامسي محقق:

$$(*) \qquad x(l) = A = B = 0 \qquad l^2 - 1 + 2al = 0$$

منطوياً نصف متناظر (أو مثيل المتناظر). نبرهن بعد ذلك أنه إذا حقق منطو متري تلامسي ثلاثـــي الأبعاد M^3 العلاقات $(*)$، فعندئذ إما $a = 0$ ، و M^3 سيكون نصف متناظر، أو $a \neq 0$ وعندئـــذ a سيكون ثابتاً، ولا يمكن لــ M^3 أن يكون نصف متناظر. وأخيراً نبرهن أنه إذا كانت قـــيم المقـــادير $x(l)$ و A و B ثابتة (وفي حالة أخرى $x(l)$ و $e(l)$ ثابتة و $j\ e(l) \neq 0$) في منطو متري تلامسي ثلاثي الأبعاد ونصف متناظر، فلابد أن يكون: $a = x(l) = A = B = 0$

الكلمات المفتاحية: المنطويات التلامسية الثلاثية وشروط التقوس.

1. Introduction

A Riemannian manifold (M, g) is called semi-symmetric if its curvature tensor R satisfies the condition

$$R(X, Y)\cdot R = 0$$

for all vector fields *X*, *Y*on M, where *R(X,Y)* acts as a derivation on R:

$$\left(R(X, Y)\cdot R\right)(U, V, W) = R(X, Y)\left(R(U, V)\, W\right) -$$
$$- R\left(R(X, Y)U, V\right)W - R\left(U, R(X, Y)V\right)W - R(U, V)\left(R(X, Y)W\right)$$

G. Calvaruso and D. Perrone in [3] have studied the semi-symmetric contact metric three-manifold with Ricci curvature $r(x, x)$ constant, which means $x(l) = 0$.

In this article we study the contact metric three-manifold M^3 satisfying certain curvature conditions especially $x(l) = const.$

2. Preliminaries

A contact metric manifold is a $(2n+1)$-dimensional differentiable manifold M^{2n+1} which carries a global differential 1-form h such that $h \wedge (dh)^n \neq 0$ everywhere on M^{2n+1}. Every contact metric manifold has an underlying contact Riemannian structure (j, x, h, g), where x is a global vector field (called the characteristic vector field or Reeb vector field), j a global tensor field of type $(1,1)$ and g a Riemannian metric (called associated metric).

These structure tensors satisfy :

$h(x) = 1,\ j^2 = -I + h \otimes x,\ h(X) = g(x, X)$

$dh(X, Y) = g(X, jY), g(jX, jY) = g(X, Y) - h(X)h(Y)$

$dh(x, X) = 0$

We denote by ∇ the Riemannian connection of g, and by R the corresponding Riemannian curvature tensor given by :

$R(X, Y) = R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$

The Ricci tensor of type $(0, 2)$, the corresponding Ricci operator and the scalar curvature are respectively indicated by r ,Q , and r . If r is constant then M^{2n+1} is said to be conformally flat .

In the theory of contact metric manifolds the tensor fields

$h = \frac{1}{2}\mathcal{L}_x\, j$ and $\ell = R(\cdot, x)x$

where $\mathcal{L}$ is the Lie derivation , play a fundamental role. On every contact metric manifold M^{2n+1} the following relations hold [4]

$$j\,x = hx = \ell\,x = 0 \qquad h\,\mathbf{o}\,j = h\,\mathbf{o}\,h = 0$$

$$\nabla_X x = -j\,X - j\,hX \qquad hj = -j\,h$$

$$\nabla_x h = j - j\,\ell - j\,h^2 \qquad j\,\ell\,j - \ell = 2(j^2 + h^2)$$

$$tr\,h = tr\,hj = 0 \qquad tr\,\bar{\ell} = g(Qx, x) = 2n - tr\,h^2$$

$$\nabla_x j = 0 \qquad \nabla_x x = 0$$

If f is a real function and the almost complex structure J on $M^{2n+1} \times R$ defined by :

$$J(Y, f\frac{d}{dt}) = (j\,Y - fx\;, h(Y)\frac{d}{dt})$$

is integrable, then the structure is said to be normal and the manifold is called Sasakian .

Let now (M^3, h, g, x, j) be a three-dimensional contact metric manifold. Let U be the open subset of M^3 where $h \neq 0$ and V the open subset of points $p \in M^3$ such that $h = 0$ in a neighborhood of p . For any point $m \in U \cup V$ there exist a local orthonormal basis $\{x, e, j\,e\}$ $([2]\,p34)$ of smooth eigen vectors of h in a neighborhood of m . On U we put $he = l\,e$ $\quad hj\,e = -l\,j\,e$

where l is a non-vanishing smooth function which is supposed to be positive .

$2-1$ **PROPOSITION** [5]

On U we have

$$\nabla_x x = 0 \quad \nabla_x e = -aj\,e \qquad \nabla_x j\,e = a\,e$$

$$(2,1) \qquad \nabla_e x = -(l+1)j\,e \qquad \nabla_e e = -mj\,e$$

$$\nabla_e j\,e = (l+1)x + m\,e$$

$$\nabla_{j\,e} x = -(l-1)e \qquad \nabla_{j\,e} e = (l-1)x + s\,j\,e$$

$$\nabla_{j\,e} j\,e = -s\,e \qquad \nabla_x h = 2ahj + x(l)s$$

where a is a smooth function ,

$$m = \frac{1}{2l}\{(j\ e)(l) + A\}, \qquad s = -\frac{1}{2l}\{e(l) + B\}$$

$A = r(x, e)$, $B = r(x, j\ e)$ and s is the $(1,1)$-tensor defined by $sx = 0$, $se = e$, $s j\ e = -j\ e$

$2-2$ **PROPOSITION** [3]

The components of the Ricci operator Q with respect to $\{x, e, j\ e\}$, are given by

$$Qx = 2(1 - l^2)x + Ae + Bj\ e$$

$$Qe = Ax + (\frac{r}{2} - 1 + l^2 + 2al)e + x(l)j\ e$$

$$Qj\ e = Bx + x(l)e + (\frac{r}{2} - 1 + l^2 - 2al)j\ e$$

From which it follows :

$$(\nabla_x Q)x = -4lx(l)x + \{x(A) + aB\}e + \{x(B) - aA\}j\ e$$

$$(\nabla_e Q)e = \{e(A) + (l+1)x(l) + Bm\}x + $$
$$+ \{e(a + 2al) + 2mx(l)\}e +$$
$$(2,2) \qquad + \{ex(l) + 2a(j\ e)(l) + (2a - l - 1)A\}j\ e$$
$$(\nabla_{j\ e} Q)j\ e = \{(j\ e)(B) + (l-1)x(l) + As\}x +$$
$$+ \{(j\ e)x(l) - 2ae(l) + (1 - l - 2a)B\}e +$$
$$+ \{(j\ e)(a - 2al) + 2sx(l)\}j\ e$$

where : $a = \frac{r}{2} - 1 + l^2$

$2-3$ **PROPOSITION** [3]

The components of R, with respect to the basis $\{x, e, j\ e\}$ are given by:

$$R(x, e)x = -(l^2 - 1 - 2al)e + x(l)j\ e$$
$$R(x, j\ e)x = x(l)e - (l^2 - 1 + 2al)j\ e$$
$$R(e, j\ e)x = -Be + Aj\ e$$
$$R(x, e)e = (l^2 - 1 - 2al)e - Bj\ e$$
$$R(x, j\ e)e = -x(l)x + Aj\ e$$

$$(2,3)\qquad \begin{aligned} R(e,j\,e)e &= Bx+(\frac{r}{2}+2l^2-2)j\,e \\ R(x,e)j\,e &= -x(l)x+Be \\ R(x,j\,e)j\,e &= (l^2-1+2al)x-Ae \\ R(e,j\,e)j\,e &= -Ax-(\frac{r}{2}+2l^2-2)e \end{aligned}$$

where :

$R(e_i,e_j)e_k=-R(e_j,e_i)e_k \quad , \qquad i,j,k=1,2,3$

$e_1=x \quad , \qquad e_2=e \quad , \qquad e_3=j\,e$

$2-4$ **PROPOSITION** [3]

Let (M^3,h,g,j,x) be a non-Sasakian contact metric three-manifold . Then M^3 is semi-symmetric if and only if

$$(2,4)\qquad \begin{aligned} B(l^2-1+2al) &= Ax(l) \\ A(l^2-1-2al) &= Bx(l) \\ AB+x(l)(\frac{r}{2}+2l^2-2) &= 0 \\ B^2-[x(l)]^2+(l^2-1-2al)(\frac{r}{2}+3l^2-3+2al) &= 0 \\ A^2-[x(l)]^2+(l^2-1+2al)(\frac{r}{2}+3l^2-3-2al) &= 0 \end{aligned}$$

3.Some classes of Semi-symmetric contact metric manifolds

$3-1$ **LEMMA**

Let (M^3,h,g,j,x) be a contact metric three-manifold . Then the following formulas hold

$$(3,1)\qquad \begin{aligned} r(x,x) &= 2(1-l^2) \\ r(e,e) &= 2al-m^2-s^2-2a-e(s)-(j\,e)(m)= \\ &= \frac{r}{2}+l^2-1+2al \\ r(j\,e,j\,e) &= -2al-m^2-s^2-2a-e(s)-(j\,e)(m)= \end{aligned}$$

$$= \frac{r}{2} + l^2 - 1 - 2al$$

$$r = 2(1 - l^2 + g)$$

where : $g = -m^2 - s^2 - 2a - e(s) - (j\,e)(m)$

Proof :

We have

$$r(x,x) = g\big(R(x,x)x,x\big) + g\big(R(x,e)x,e\big) + g\big(R(x,j\,e)\,x,j\,e\big)$$

And if we use $(2,3)$ we obtain:

$$r(x,x) = 2(1 - l^2)$$

We have also :

$$r(e,e) = g\big(R(e,x)e,x\big) + g\big(R(e,e)e,e\big) + g\big(R(e,j\,e)\,e,j\,e\big)$$

And if we use $(2,1)$ and $(2,3)$ we find :

$$g\big(R(e,x)e,x\big) = -(l^2 - 1 - 2al)$$

$$R(e,j\,e)\,e = \nabla_{[e,j\,e]}e - \nabla_e\nabla_{j\,e}e + \nabla_{j\,e}\nabla_e e$$

$$= \nabla_{2x+me-sj\,e}e - \nabla_e\big((l-1)x + sj\,e\big) + \nabla_{j\,e}(-mj\,e)$$

$$= [-2ls - e(l)]x + \big\{l^2 - 1 - m^2 - s^2 - 2a - e(s) - (j\,e)(m)\big\}j\,e$$

$$= Bx + \big\{l^2 - 1 - m^2 - s^2 - 2a - e(s) - (j\,e)(m)\big\}j\,e$$

And then :

$$r(e,e) = 2al - m^2 - s^2 - 2a - e(s) - (j\,e)(m)$$

Similarly we obtain the value of $r(j\,e,j\,e)$, and hence we have:

$$r = r(x,x) + r(e,e) + r(j\,e,j\,e)$$

$$= 2\big\{(1 - l^2) - m^2 - s^2 - 2a - e(s) - (j\,e)(m)\big\}$$

Now it follows easily from $(2,4)$ and $(3,1)$:

$3-2$ **COROLLARY**

Let (M^3, h, g, j, x) be a contact metric three-manifold satisfying

$$x(l) = A = B = 0,\quad l^2 - 1 + 2al = 0$$

$$l^2 - 1 - 2al \neq 0$$

Then M^3 is semi-symmetric if and only if one of the following conditions is satisfied :

(I) $r = 4(1 - l^2)$

(II) $r = 8al$

(III) $r(j\,e, j\,e) = 0$

(IV) $r(e, e) = 4al$

3 − 3 **COROLLARY**

Let (M^3, h, g, j, x) be a contact metric three-manifold satisfying

$R(x, j\,e)\,x = 0, \quad R(e, j\,e)\,x = 0, \quad R(e, j\,e)\,j\,e = 0$

Then M^3 is semi-symmetric .

3 − 4 PROPOSITION

Let (M^3, h, g, j, x) be a contact metric three-manifold satisfying

$$(3-2) \qquad x(l) = 0, \quad A = B = 0, \quad l^2 - 1 + 2al = 0$$

Then either $a = 0$ and M^3 is semi-symmetric or $a \neq 0$. In this case a is constant , $r = 4a(l - 1)$ and M^3 cannot be semi-symmetric .

Proof .

Differentiating $l^2 - 1 + 2al = 0$ with respect to x we get $x(a) = 0$. Next differentiating with respect to e , we obtain:

$$(3-3) \qquad l\big(e(l) + e(a)\big) = -a\,e(l)$$

We have the well-known formula

$$(3-4) \qquad \frac{1}{2}X(r) = \sum_{i=1}^{n} g((\nabla_{e_i} Q)e_i, X)$$

which holds for any vector field X of n-dimensional Riemannian manifold where $\{e_i\}$ is an arbitrary basis .

If we put $X = e$, $e_1 = x$, $e_2 = e$, $e_3 = j\,e$

and using $(2,2)$ and $(3,2)$ we obtain :

$$\frac{1}{2}e(r) = g\big((\nabla_x Q)x, e\big) + g\big((\nabla_e Q)e, e\big) + g\big((\nabla_{j\,e} Q)\,j\,e, e\big)$$

$$= \frac{1}{2}e(r) + 2l\,[e(l) + e(a)]$$

Since $l \neq 0$, it follows :

$e(l) + e(a) = 0$

Using $(3,3)$ we obtain :

$ae(l)=0$

Then either $a=0$, and from $(3,2)$ we find that $l=1$ and hence all the relations $(2,4)$ are satisfied ,which means that M^3 is semi-symmetric, or $a \neq 0$, $e(l)=0$ and then also $e(a)=0$.

If we apply $(3,4)$ and $(2,2)$ to compute $\frac{1}{2} j\ e(r)$ we obtain :

$(j\ e)(l)=(j\ e)(a)$

Differentiating the last relation of $(2,2)$ with respect to $j\ e$ and using the last equation we find: $j\,(e)(a)\cdot(2l+a)=0$, and since $a=\frac{1-l^2}{2l}$, the last formula becomes:

$(j\ e)(a)\cdot(3l^2+1)=0$

Thus $(j\ e)(a)=0$ and so $(j\ e)(l)=0$

Now from

$x(a)=e(a)=(j\ e)(a)=0$

$x(l)=e(l)=(j\ e)(l)=0$

We can conclude that a and l are constants .

Using $A=B=m=s=0$ and from the relations $(3,1)$ we find :

$r(x,x)=2(1-l^2)$

$r(e,e)=2a(l-1)$

$r(j\ e, j\ e)=-2a(l+1)$

$r=4a(l-1)$

Finally if we assume that M^3 is semi-symmetric then according to lemma $3,3$ of [3] we have $a=0$ which contradicts our assumption .

$3-5$ **THEOREM**

Let M^3 be a semi-symmetric contact metric three-manifold satisfying :

(3,5) $\quad x(l) = const.$, $\quad A = const.$, $\quad B = const.$

Then $\quad a = x(l) = A = B = 0$

and M^3 is either flat or sasakian .

<u>Proof</u>

We multiply the first relation of $(2,4)$ by B , and the second by A , then we subtract the first product from the second product and we get

$$B^2(l^2 - 1 + 2al) - A^2(l^2 - 1 - 2al) = 0$$

Using $(2,4)$ to express B^2 and A^2 we find

(3,6) $$4al\left\{(l^2-1)^2 - 4a^2l^2 - [x(l)]^2\right\} = 0$$

Then either $a = 0$ and we shall study the case, or $a \neq 0$ and we shall prove that this case can not occur.

If $a = 0$, then the first, second, fourth and fifth relations of $(2,4)$ become :

$$B(l^2-1) = Ax(l)$$

$$A(l^2-1) = Bx(l)$$

$$B^2 - [x(l)]^2 + (l^2-1)(\frac{r}{2} + 3l^2 - 3) = 0$$

$$A^2 - [x(l)]^2 + (l^2-1)(\frac{r}{2} + 3l^2 - 3) = 0$$

From which it follow A= ±B, and the relations (2,4) are equivalent to :

$$A(l^2-1) = \pm Ax(l)$$

(3,7) $$\pm A^2 + x(l)(\frac{r}{2} + 2l^2 - 2) = 0$$

$$A^2 = B^2 = [x(l)]^2 + (l^2-1)(\frac{r}{2} + 3l^2 - 3)$$

If $A = 0$ then [3]

$$x(l) = A = B = a = 0$$

and M^3 either is flat or has constant curvature 1 .

We shall prove that the case $A \neq 0$ can not occur .

In fact consider $U_1 = \{p \in M^3 / A \neq 0 \text{ at } p\}$. Then from the first relation of $(3,7)$ we get on U_1

$const. = x(l) = \pm(l^2 - 1)$

from which it follows that $l = const.$ and hence $x(l) = 0$, and the second relation of $(3,7)$ now yields $A = 0$ on U_1 which can not occur . Therefore , U_1 is empty , and $A = 0$ on M^3.

We assume now $a \neq 0$. Differentiating the third relation of $(2,4)$ with respect to x we get

$$x(l)\left[\frac{x(r)}{2} + 4l\,x(l)\right] = 0$$

If $x(l) = 0$ on some neighbourhood U_2 , then $a = 0$ [3] , which contradicts our assumption .

Let $x(l) \neq 0$ and $\frac{x(r)}{2} + 4l\,x(l) = 0$.

Then:

$$(3,8) \qquad x(r) = -8l\,x(l)$$

From $(3,6)$ we find :

$$(3,9) \qquad [x(l)]^2 = (l^2 - 1)^2 - 4a^2 l^2$$

We add the fourth and fifth relations of $(2,4)$, and we use $(3,9)$ to express $[x(l)]^2$.

We obtain :

$$(3,10) \qquad A^2 + B^2 = r(1 - l^2) - 4(1 - l^2)^2$$

Differentiating $(3,10)$ with respect to x we find :

$$(1 - l^2)x(l) - 2rl\,x(l) - 8(1 - l^2)(-2l)x(l) = 0$$

We use $(3,8)$ to express $x(r)$.

We get :

$$r = 4(1 - l^2)$$

From $(3,10)$ it follows that: $A^2 + B^2 = 0$

and hence $A = B = 0$ and then $a = 0$ which contradicts our assumption .

$3-6$ **PROPOSITION**

Let M^3 be a semi-symmetric contact metric three-manifold satisfying :

$x(l) = const.,\qquad e(l) = const.,\ j\ e(l) \neq 0$

Then $a = x(l) = A = B = 0$ and M^3 is either flat or Sasakian .

Proof

On M^3 we have the relation $(3,6)$

$4al\left\{(l^2-1)^2 - 4a^2l^2 - [x(l)]^2\right\} = 0$

If $a = 0$ the conclusion follows as in the proof of theorem $3-5$. We shall prove that the case $a \neq 0$ can not occur .

In fact consider $U_3 = \left\{p \in M^3 / a \neq 0 \text{ at } p\right\}$. On U_3 we have :

(3,11) $\qquad [x(l)]^2 = (l^2-1)^2 - 4a^2l^2$

On the other hand , using $(2,1)$ we have:

$[x,e](l) = x\,e(l) - ex(l) = (\nabla_x e - \nabla_e x)(l)$

Therefore :

$(l+1-a)(j\ e)(l) = 0$

and so : $a = l+1$. then $(3,11)$ becomes

$[x(l)]^2 = (l^2-1)^2 - 4l^2(l+1)^2$

Differentiating with respect to x we obtain :

$(-12l^3 - 24l^2 - 12l)x(l) = 0$

It is now easy to conclude that $x(l) = 0$. In fact if we assume $x(l) \neq 0$, then $(-12l^3 - 24l^2 - 12l) = 0$, and differentiating with respect to x we get $(-36l^2 - 48l - 12)x(l) = 0$ and if we continue in the same way we find at last that $x(l) = 0$, which contradicts our assumption . Hence $x(l) = 0$, and then $a = 0$ on U_3, which can not occur .

Therefore U_3 is empty and $a = 0$ on M^3 .

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