Developing Hilbert's Nullstellensatz into Categorical Adjunction

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ABSTRACT

This paper includes an improvement of the categorical isomorphism between the category of radical ideals and the category of affine algebraic sets into an adjunction between two functors. Afterwards, we extend the previous functors by means of expanding our work into the category of all ideals in a polynomial ring in *n* variables over a fixed algebraically closed field *k*, in which the radical ideals form a full subcategory of it, in order to produce more generalized adjunction from which we obtain the following results:

- $\phi^{\chi} : \underline{AS}^{[n,k]}(\mathbb{V}-,-) \dot{\rightarrow} \underline{Id}^{[n,k]}(-,\mathbb{I}-)$.
- $\varepsilon\mathbb{V} \circ \mathbb{V}\chi = \mathbb{1}_{\mathbb{V}}$.
- $\chi\mathbb{I} \circ \mathbb{I}\varepsilon = \mathbb{1}_{\mathbb{I}}$.

which illustrates a development of Hilbert's Nullstellen-satz.

Keywords: Adjunction, Adjunction unit, Co-unit, Categorical isomorphism, Radical ideals.

Mathematical subject classification (MSC 2010): 18A40,16B50,18F99

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تَطويرُ نَظريَّة Hilbert للأصفَارِ إلى تَرافُقٍ فِئَويٍ

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الملخص

تحتوي هذه الورقة على تطوير للتماثل بين فئتي المثاليات الجذرية والمجموعات الجبرية التآلفية إلى ترافق فئوي بين تشاكلين فئويين. ومُدّد مفهوما التشاكلين الفئويين السابقين من خلال توسيع العمل إلى فئة المثاليات في حلقة حدوديات بـــ n متغيراً فوق حقل مثبت مغلق جبرياً k التـــي تـــشكل المثاليـــات الجذرية فئة جزئية ممتلئة منها وذلك للحصول على ترافق أعم حصلنا من خلاله على النتائج الآتية:

- $\phi^{\chi} : \underset{-}{\overset{[n,k]}{AS}}(\mathbb{V}-,-) \dot{\to} \underset{-}{\overset{[n,k]}{Id}}(-,\mathbb{I}-)$.
- $\varepsilon\mathbb{V} \circ \mathbb{V}\chi = \mathbf{1}_{\mathbb{V}}$.
- $\chi\mathbb{I} \circ \mathbb{I}\varepsilon = \mathbf{1}_{\mathbb{I}}$.

التي تعدُّ تطويراً لنظرية Hilbert للأصفار.

الكلمات المفتاحية: ترافق، نواة ترافق، مرافق النواة، تماثل فئوي، مثالي جذري.

التصنيف الرياضي العالمي (MSC: 2010): 18A40, 16B50, 18F99

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Introduction

Hilbert's Nullstellensatz reveals essential relations between affine algebraic sets in n-affine space $\mathbb{A}_k^n$, whereas k is assumed to be an algebraically closed field unless stated otherwise, and the radical ideals in a polynomial ring in n varibles over the fixed field k. This theory has been developed in [10] using category theory into the following categorical isomorphism:

$$\overset{[n,k]}{\underline{AS}} \approx \overset{[n,k]}{\underline{RI}}$$

where the former is the category of all radical ideals in $k[x_1, \cdots, x_n]$ and the latter is the category of all possible affine algebraic sets in $\mathbb{A}_k^n$. The previous result was deduced through extending the concept of both algebraic set of an ideal and the ideal of an algebraic set into the following two contravariant functors, respectively:

$$\mathbb{I}: \overset{[n,k]}{\underline{AS}} \longrightarrow \overset{[n,k]}{\underline{RI}}\ , \qquad \mathbb{V}: \overset{[n,k]}{\underline{RI}} \longrightarrow \overset{[n,k]}{\underline{AS}}$$

In this paper, we prove that both $\mathbb{I}$ and $\mathbb{V}$ form an adjunction in order to develop the isomorphism in [10] into the following adjunction:

$$\overset{[n,k]}{\underline{AS}} \rightharpoonup \overset{[n,k]}{\underline{RI}}$$

which completes the first step. As for the next step, we extend the above adjunction into the category of all possible ideals in $k[x_1, \cdots, x_n]$, from which $\overset{[n,k]}{\underline{RI}}$ is proved to be a full subcategory, in [10], using a natural transformation shown in [10].

1-Imperative Definitions:

Definition 1.1: [4][5][8][9][13][14]

Let k be a fixed algebraically closed field. We define affine n-space over k, denoted $\mathbb{A}_k^n$ or simply $\mathbb{A}^n$, to be the set of all n-tuples of elements of k. An element $P \in \mathbb{A}_k^n$ will be called a point.

If $P = (a_1, \cdots, a_n)$ with $a_i \in k$, then the a_i will be called the coordinates of P.

Definition 1.2: [4][5][8][9][13][14]

A subset V of $\mathbb{A}_k^n$ is an algebraic set if there exists a subset $T \subseteq k[x_1, \cdots, x_n]$ such that:

$$V = \mathbb{V}(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T\}$$

Clearly, if I is the ideal of $k[x_1, \cdots, x_n]$ generated by T, then $\mathbb{V}(I) = \mathbb{V}(T)$. Furthermore, since $k[x_1, \cdots, x_n]$ is a noetherian ring, any ideal I has a finite set of generators $f_1, \cdots, f_r$. Thus, $\mathbb{V}(T)$ can be expressed as the common zeroes of the finite set of the polynomials $f_1, \cdots, f_r$.

Proposition 1.3: [4][5][8][9]

The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.

Definition 1.4: [4][5][8][9]

A nonempty algebraic subset V of $\mathbb{A}_k^n$ is said to be irreducible if it cannot be expressed as the union $V = V_1 \cup V_2$ of two proper algebraic subsets. The empty set is not considered to be irreducible.

Definition 1.5: [4][5][8][9][13][14]

For any set X of $\mathbb{A}_k^n$, we consider those polynomials that vanish on X; they form an ideal in $k[x_1, \cdots, x_n]$ called the ideal of X and written $\mathbb{I}(X)$:

$$\mathbb{I}(X) = \{f \in k[x_1, \cdots, x_n] \mid f(P) = 0 \text{ for all } P \in X\}$$

Proposition 1.6: [4][5][9][13]

An algebraic set V is irreducible if and only if $\mathbb{I}(V)$ is prime.

Proposition 1.7: [4][5][9][13][14]

If I is a proper ideal in $k[x_1, \cdots, x_n]$ then $\mathbb{V}(I) \neq \emptyset$.

Definition 1.8: [6][9][15]

For any arbitrary ideal I in a commutative ring R, we define the radical ideal of I, which is denoted by $\operatorname{Rad} I$ or $\sqrt{I}$, as follows:

$$\operatorname{Rad} I = \{a \in R \mid \exists n \in \mathbb{N}, a^n \in I\}$$

Proposition 1.9: [4][5][6][8][9][13][14][15]
If $f_1, \cdots, f_r$ and g are in $k[x_1, \cdots, x_n]$ where k is algebraically closed field, and g vanishes whenever $f_1, \cdots, f_r$ vanish, then there is an equation:

$$g^N = h_1 f_1 + \cdots + h_n f_n$$

for some $N > 0$ and some $h_i \in k[x_1, \cdots, x_n]$.

In concrete terms, this says the following: Let I be an ideal in $k[x_1, \cdots, x_n]$ (k is algebraically closed), then $\mathbb{I}(\mathbb{V}(I)) = \text{Rad}\, I$.

The previous section was all about basic concepts in algebraic geometry. Also, we note that propositions 1.7 and 1.9 are called the weak nullsellensatz and Hilbert's nullstellensatz, respectively.
Now we set some introductory definitions and propositions in category theory:

Definition 1.10: [1][2][3][7][11][12][15]
A category $\underline{C}$ consists of two sets denoted by A, O which are called the set of arrows and the set of objects, respectively, with two functions:

$$A \underset{\text{cod}}{\overset{\text{dom}}{\rightrightarrows}} O$$

In this diagram scheme the set of composable pairs of arrows is:

$$A \times_O A = \{\langle g, f \rangle \mid g, f \in A \text{ and } \text{dom}\, g = \text{cod}\, f\}$$

which is called the product over O.

The previous diagram scheme is equipped with two additional functions called "identity" and "composition" as follows:

$$O \xrightarrow{\text{Id}} A \quad , \quad A \times_O A \xrightarrow{\circ} A$$
$$c \mapsto \text{Id}\, c \qquad \langle g, f \rangle \mapsto g \circ f$$

such that:

$$\text{dom}(\text{Id}\, a) = a = \text{cod}(\text{Id}\, a)\ , \qquad \begin{array}{c} \text{dom}(g \circ f) = \text{dom} f \\ \text{cod}(g \circ f) = \text{cod}\, g \end{array}$$

in which the following associative and unit laws, respectively, hold:

$$h \circ (g \circ f) = (h \circ g) \circ f\ , \qquad \begin{array}{c} \text{Id}(\text{cod} f) \circ f = f \\ f \circ \text{Id}(\text{dom} f) = f \end{array}$$

Definition 1.11: [1][2][3][7][11][12][15]
For two categories $\underline{C}$, $\underline{B}$, a functor $T:\underline{C}\longrightarrow\underline{B}$ consists of two suitably related functions: the object function T which assigns to each object $c\in\underline{C}$ an object $Tc\in\underline{B}$ and the arrow function (also written T) which assigns to each arrow $f:c\longrightarrow c'$ of $\underline{C}$ an arrow $Tf:Tc\longrightarrow Tc'$ of $\underline{B}$, in such a way that:

$$T(\mathrm{Id}\,c)=\mathrm{Id}(Tc)\,,\qquad T(g\circ f)=Tg\circ Tf$$

the latter whenever the composite $g\circ f$ is defined in $\underline{C}$.

Definition 1.12: [1][2][3][12][15]
Given two functors $S,T:\underline{C}\longrightarrow\underline{B}$, a natural transformation $\tau:S\dot{\longrightarrow}T$ is a function which assigns to each object $c\in\underline{C}$ an arrow $\tau_c:Sc\longrightarrow Tc$ in $\underline{B}$ in such a way that every arrow $f:c\longrightarrow c'$ in $\underline{C}$ satisfies the following:

$$\tau_{c'}\circ Sf=Tf\circ\tau_c$$

When this holds we say that $\tau_c:Sc\longrightarrow Tc$ is natural in c.

Definition 1.13: [2][3][7]
For a category $\underline{C}$, we define the hom-set of objects $a,b\in\underline{C}$ consists of all arrows of the category with domain a and codomain b as follows:

$$\underline{C}(a,b)=\{f\text{ in }\underline{C}\mid \mathrm{dom}\,f=a\text{ and }\mathrm{cod}\,f=b\}$$

Definition 1.14: [3][7][11][14]
If $S:\underline{D}\longrightarrow\underline{C}$ is a functor and c an object of $\underline{C}$, a universal arrow form c to S is a pair $\langle r,u\rangle$ consisting of an object r of $\underline{D}$ and an arrow $u:c\longrightarrow Sr$ of $\underline{C}$, such that to every pair $\langle d,f\rangle$ with d an object of $\underline{D}$ and $f:c\longrightarrow Sd$ an arrow of $\underline{C}$, there exists a unique arrow $f':r\longrightarrow d$ of $\underline{D}$ with $Sf\circ u=f'$. In other words, every arrow f to S factors uniquely through the universal arrow u.

Definition 1.15: [3][12]
Let $\underline{A}$ and $\underline{X}$ be categories. An adjunction from $\underline{X}$ to $\underline{A}$ is a triple $\langle F,G,\phi\rangle:\underline{X}\rightharpoonup\underline{A}$, where F and G are two functors:

$$\underline{X}\overset{F}{\underset{G}{\rightleftarrows}}\underline{A}$$

while ϕ is a function which assigns to each pair of objects $x \in \underline{X}$, $a \in \underline{A}$ a bijection of sets:

$$\phi_{x,a}: \underline{A}(Fx,a) \cong \underline{X}(x,Ga)$$

which is natural in x and a.

Theorem 1.16: [3][12]

Each adjunction $\langle F,G,\phi\rangle: \underline{X} \rightharpoonup \underline{A}$ is completely determined by the functors $F,G,$ and a natural transformation $\eta: \mathbf{I}_{\underline{X}} \dot{\rightarrow} G \circ F$ such that each component η_x is universal from x to G.

2- Constructing The Adjunction $\overset{[n,k]}{\underline{AS}} \rightharpoonup \overset{[n,k]}{\underline{RI}}$:

2.1 Constructing ϕ:

Let $I \subset k[x_1,\cdots,x_n]$ be a proper radical ideal, $V \subseteq \mathbb{A}_k^n$ an affine algebraic set then we have the following equivalence [10]:

$$(\mathbb{V}(I) \subseteq V) \Leftrightarrow \big(\mathbb{I}(V) \subseteq \mathbb{I}(\mathbb{V}(I))\big) \Leftrightarrow (\mathbb{I}(V) \subseteq \mathrm{Rad}\, I = I)$$

Therefore,

$$(\mathbb{V}(I) \subseteq V) \overset{*}{\Leftrightarrow} (\mathbb{I}(V) \subseteq I)$$

Now, based on the previous equivalence, we can construct the function ϕ mapping each ordered pair $\langle I,V\rangle$ to the following bijection:

$$\phi_{I,V}: \overset{[n,k]}{\underline{AS}}(\mathbb{V}(I),V) \overset{\sim}{\longrightarrow} \overset{[n,k]}{\underline{RI}}\big(I,\mathbb{I}(V)\big)$$

In order to show that $\phi_{I,V}$ is actually a bijection we will distinguish two different cases depending on the hom-sets in the preorder categories $\overset{[n,k]}{\underline{AS}}, \overset{[n,k]}{\underline{RI}}$:

1) If $\mathbb{V}(I) \nsubseteq V$ then $\overset{[n,k]}{\underline{AS}}(\mathbb{V}(I),V)$ is empty which is the case for $\overset{[n,k]}{\underline{RI}}\big(I,\mathbb{I}(V)\big)$ because $\mathbb{I}(V) \nsubseteq I$. Thus,

$$\phi_{I,V} = \emptyset: \overset{[n,k]}{\underline{AS}}(\mathbb{V}(I),V) = \emptyset \longrightarrow \overset{[n,k]}{\underline{RI}}\big(I,\mathbb{I}(V)\big) = \emptyset$$

is the empty function which is trivially a bijection in this case.

2) If $\mathbb{V}(I) \subseteq V$ then the hom-set $\overset{[n,k]}{\underline{AS}}(\mathbb{V}(I),V)$ is a singleton, so does $\overset{[n,k]}{\underline{RI}}(I,\mathbb{I}(V))$, depending on the * equivalence above, which means that $\phi_{I,V}$ in this case is well

defined and translates the unique arrow in the former hom-set to that in the latter:

$$\phi_{I,V}(\mathbb{V}(I) \hookrightarrow V) = (I \hookrightarrow \mathbb{I}(V))$$

Due to such construction, $\phi_{I,V}$ is again a bijection for this case.

2.2 Naturality of ϕ:

The previous matter is not enough to make the triple $\langle \mathbb{V},\mathbb{I},\phi\rangle: \overset{[n,k]}{\underline{AS}} \rightharpoonup \overset{[n,k]}{\underline{RI}}$ an adjunction so far, merely because this structure lacks the proof of naturality of ϕ.

Let's start by proving that ϕ is natural in the first component i.e. in a radical ideal I. In other words, for every arrow $\ddot{I} \hookrightarrow \dot{I}$, the following diagram will be commutative:

$$\begin{array}{ccccc}
\dot{I} & & \overset{[n,k]}{\underline{AS}}(\mathbb{V}(\dot{I}),V) & \xrightarrow{\ \phi_{I,V}\ } & \overset{[n,k]}{\underline{RI}}(\dot{I},\mathbb{I}(V)) \\
\uparrow\!\cup & ; & \# \quad \uparrow & & \downarrow \ * \\
\ddot{I} & & \overset{[n,k]}{\underline{AS}}(\mathbb{V}(\ddot{I}),V) & \xrightarrow[\ \phi_{I,V}\ ]{} & \overset{[n,k]}{\underline{RI}}(\ddot{I},\mathbb{I}(V))
\end{array}$$

Again we'll apply the same technique used before where we distinguish two cases.

If $\mathbb{V}(\dot{I}) \nsubseteq V$, then we have $\ddot{I} \nsubseteq \mathbb{I}(V)$, $(\mathbb{V}(\dot{I}) \subseteq V(\ddot{I})$ and $\mathbb{V}(\dot{I}) \nsubseteq V$ gives $\mathbb{V}(\ddot{I}) \nsubseteq V$. Thus, $\mathbb{I}(V) \nsubseteq \mathbb{I}(V(\ddot{I}))$ = Rad T which gives $\mathbb{I}(V) \nsubseteq (\ddot{I}))$ by means of *, which gives again $\mathbb{V}(\ddot{I}) \nsubseteq V$. Therefore, all hom-sets in the diagram are empty as for all the

functions in between which gives, trivially, the aspect of commutativity in this case.
Now, for the second case where $\mathbb{V}(\dot{I}) \subseteq V$ holds, one can easily deduce, using the equivalence *, that all hom-sets in the diagram are singletons for which we have the following:

$$
\begin{aligned}
(\star \circ \phi_{I,V} \circ \#)(\mathbb{V}(\ddot{I}) \hookrightarrow V) &= \star\Big(\phi_{I,V}\big(\#(\mathbb{V}(\ddot{I}) \hookrightarrow V)\big)\Big) \\
&= \star\Big(\phi_{I,V}\big(\mathbb{V}(\dot{I}) \hookrightarrow (\mathbb{V}(\ddot{I}) \hookrightarrow V)\big)\Big) \\
&= \star\Big(\phi_{I,V}\big(\mathbb{V}(\dot{I}) \hookrightarrow V\big)\Big) = \star\big(\dot{I} \hookrightarrow \mathbb{I}(V)\big) \\
&= \ddot{I} \hookrightarrow \dot{I} \hookrightarrow \mathbb{I}(V) = \ddot{I} \hookrightarrow \mathbb{I}(V) \\
&= \phi_{I,V}\big(\mathbb{V}(\ddot{I}) \hookrightarrow V\big)
\end{aligned}
$$

This gives $\star \circ \phi_{I,V} \circ \# = \phi_{I,V}$ which leads to commutativity for this case.
Let's, now, continue towards showing that ϕ is natural in the second component i.e. in some affine algebraic set V. In order to do so, we must show that for every arbitrary arrow $\dot{V} \hookrightarrow \ddot{V}$ the following diagram will be commutative:

$$
\begin{array}{ccccc}
\dot{V} & & \underset{}{\overset{[n,k]}{\underline{AS}}}\big(\mathbb{V}(I),\dot{V}\big) & \xrightarrow{\phi_{I,\dot{V}}} & \overset{[n,k]}{\underline{RI}}\big(I,\mathbb{I}(\dot{V})\big) \\
\cap\downarrow & ; & \#\downarrow & & \uparrow * \\
\ddot{V} & & \overset{[n,k]}{\underline{AS}}\big(\mathbb{V}(I),\ddot{V}\big) & \xrightarrow[\phi_{I,\ddot{V}}]{} & \overset{[n,k]}{\underline{RI}}\big(I,\mathbb{I}(\ddot{V})\big)
\end{array}
$$

Let's first discuss the case where $\mathbb{V}(I) \not\subseteq \dot{V}$ then $I \not\subseteq \mathbb{I}(\dot{V})$ also $I \not\subseteq \mathbb{I}(\ddot{V})$, because $\mathbb{I}(\ddot{V}) \subseteq \mathbb{I}(\dot{V})$, [$\dot{V} \subseteq \ddot{V}$ gives $\mathbb{I}(\ddot{V}) \subseteq \mathbb{I}(\dot{V})$ and $I \not\subseteq \mathbb{I}(\dot{V})$. Thus, $I \not\subseteq \mathbb{I}(\ddot{V})$] and this means that $\mathbb{V}(I) \not\subseteq \ddot{V}$ which confirms that all hom-sets are empty. Thus, all functions are empty, in the diagram, in a way that makes it trivially commutative regarding this case.

Now, for the case where $\mathbb{V}(I) \subseteq \dot{V}$, it's easy to see that all related hom-sets are singletons. Therefore, we have:

$$
\begin{aligned}
(\star \circ \phi_{I,\ddot{V}} \circ \#)(\mathbb{V}(I) \hookrightarrow \dot{V}) &= \star\left(\phi_{I,\ddot{V}}\left(\#(\mathbb{V}(I) \hookrightarrow \dot{V})\right)\right)\\
&= \star\left(\phi_{I,\ddot{V}}(\mathbb{V}(I) \hookrightarrow \dot{V} \hookrightarrow \ddot{V})\right)\\
&= \star\left(\phi_{I,\ddot{V}}(\mathbb{V}(I) \hookrightarrow \ddot{V})\right) = \star\left(I \hookrightarrow \mathbb{I}(\ddot{V})\right)\\
&= I \hookrightarrow \mathbb{I}(\ddot{V}) \hookrightarrow \mathbb{I}(\dot{V}) = I \hookrightarrow \mathbb{I}(\dot{V})\\
&= \phi_{I,\dot{V}}(\mathbb{V}(I) \hookrightarrow \dot{V})
\end{aligned}
$$

This shows that $\star \circ \phi_{I,\ddot{V}} \circ \# = \phi_{I,\dot{V}}$ and that's what is required to prove the commutativity of the diagram.

According to what's stated above we deduce that the triple $\langle \mathbb{V}, \mathbb{I}, \phi \rangle$ forms an adjunction from the category of radical ideals in $k[x_1, \cdots, x_n]$ to the category of affine algebraic sets in $\mathbb{A}_k^n$.

2.3 Unit and Counit of $\underline{AS}^{[n,k]} \rightharpoonup \underline{RI}^{[n,k]}$:

Due to the categorical isomorphism in [10] we can deduce that the unit of the adjunction is the following identity transformation:

$$\left(\eta = \mathbb{1}_{\mathbb{I}} : \mathbb{1}_{\underline{RI}^{[n,k]}} \dot{\longrightarrow} \mathbb{I} \circ \mathbb{V} = \mathbb{1}_{\underline{RI}^{[n,k]}}\right) = (\eta_I : I = I)_{\forall I \in \underline{RI}^{[n,k]}}$$

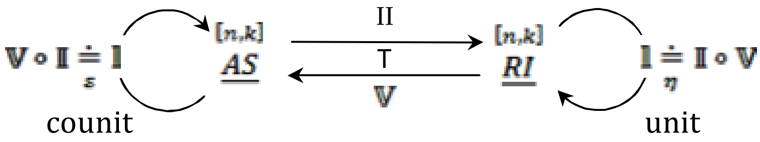
in which every component $\mathbb{1}_{\mathbb{I}} I : I = I$ is a universal arrow form I to the functor $\mathbb{I}$.

On the other hand, the same goes for the counit:

$$\left(\varepsilon = \mathbb{1}_{\mathbb{V}} : \mathbb{1}_{\underline{AS}^{[n,k]}} \dot{\longrightarrow} \mathbb{V} \circ \mathbb{I} = \mathbb{1}_{\underline{AS}^{[n,k]}}\right) = (\varepsilon_V : V = V)_{\forall V \in \underline{AS}^{[n,k]}}$$

and again every component $\mathbb{1}_{\mathbb{V}} V : V = V$ of which is a couniversal arrow form affine algebraic set V to the functor $\mathbb{V}$.

Finally, we conclude this section with the following diagram:



3-Extending into The Form $\overset{[n,k]}{\underline{AS}} \rightharpoonup \overset{[n,k]}{\underline{Id}}$:

Let's start by extending both $\mathbb{I}$ and $\mathbb{V}$ contravariant functors in the following manner, respectively:

$$\mathbb{I}: \overset{[n,k]}{\underline{AS}} \overset{\mathbb{I}}{\longrightarrow} \overset{[n,k]}{\underline{RI}} \overset{In}{\longrightarrow} \overset{[n,k]}{\underline{Id}}\ , \qquad \mathbb{V}: \overset{[n,k]}{\underline{Id}} \overset{\mathrm{Rad}}{\longrightarrow} \overset{[n,k]}{\underline{RI}} \overset{\mathbb{V}}{\longrightarrow} \overset{[n,k]}{\underline{AS}}$$

The following work is now dedicated to extend the previous adjunction given before into a new one which has the following form:

$$\langle \mathbb{V}, \mathbb{I}, \phi \rangle: \overset{[n,k]}{\underline{AS}} \rightharpoonup \overset{[n,k]}{\underline{Id}}$$

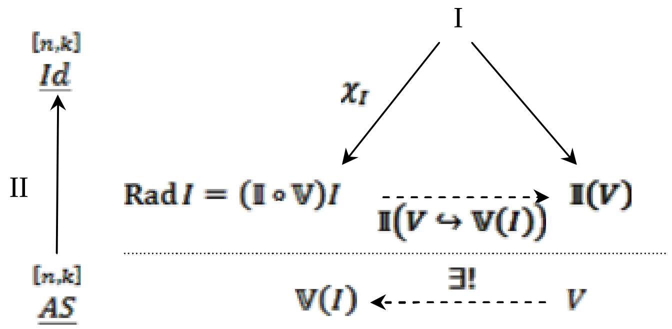
At this point, we face certain difficulties with constructing the function ϕ which gives the bijection $\phi_{I,V}$ for every ideal I, not necessarily radical, and every affine algebraic set V. The reason for that is the lack of ability to use Hilbert's nullstellensats merely because applying, it in this case, does not necessarily make $\phi_{I,V}$ a bijection! Thus, we'll apply different approach in order to do so by constructing a suitable natural transformation in a way which formalize an anticipated unit in the following section:

3.1 Establishing the Unit:

In [10], the following natural transformation was created:

$$\left(\chi: \mathbb{1}_{\overset{[n,k]}{\underline{Id}}} \dot{\longrightarrow} \mathbb{I}\circ\mathbb{V}\right) = \left(\chi_I: I \hookrightarrow (\mathbb{I}\circ\mathbb{V})I = \mathrm{Rad}\, I\right)_{\forall I \in \overset{[n,k]}{\underline{Id}}}$$

Now, if V is an arbitrary affine algebraic set in $\mathbb{A}_k^n$ such that $I \subseteq \mathbb{I}(V)$ then, we have $\mathbb{V}(\mathbb{I}(V)) \subseteq \mathbb{V}(I)$. Thus, $V \subseteq \mathbb{V}(I)$, which leads to the following diagram:



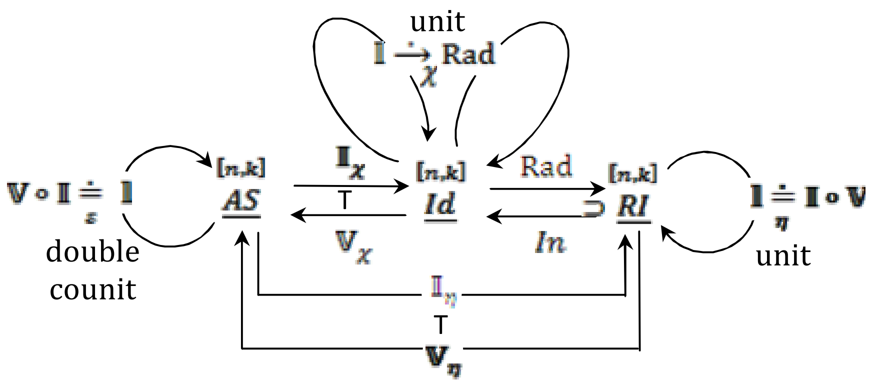
with a universal property: for all arrows $I \hookrightarrow \mathbb{I}(V)$ in the category of all ideals, whereas V is any affine algebraic set, there exists a unique arrow $V \hookrightarrow \mathbb{V}(I)$ in the category of affine sets which completes the commutativity of the diagram above i.e. the following holds:

$$\mathbb{I}(V \hookrightarrow \mathbb{V}(I)) \circ \chi_I = (\mathrm{Rad}\, I \hookrightarrow \mathbb{I}(V)) \circ \chi_I = (I \hookrightarrow \mathbb{I}(V))$$

What is stated above confirms that the pair $\langle V, \chi_I : I \hookrightarrow \mathrm{Rad}\, I \rangle$, for all ideals I, is universal form I to the contravariant functor $\mathbb{I}$.

3.2 Towards the extended adjunction:

Now that every component of the natural transformation χ is a universal arrow as stated in the previous section, it is evident that the triple $\langle \mathbb{V}, \mathbb{I}, \chi \rangle$ formalizes an adjunction from the category of ideals in $k[x_1, \cdots, x_n]$ to the category of affine algebraic sets in the *n*-affine space $\mathbb{A}_k^n$ which is shown in the following diagram of categories and functors and natural transformations that contributed in our construction:



It’s obvious that we established two pairs of adjoint functors which are distinguished, in the diagram above, using the unit as subscript where $\mathbb{V}_\eta \dashv \mathbb{I}_\eta$ is an adjunction arising from the following categorical isomorphism:

$$\overset{[n,k]}{\underline{AS}} \approx \overset{[n,k]}{\underline{RI}}$$

while $\mathbb{V}_\chi \dashv \mathbb{I}_\chi$ is the extended adjunction.

4-Consequent results:

4.1 The function ϕ^χ:

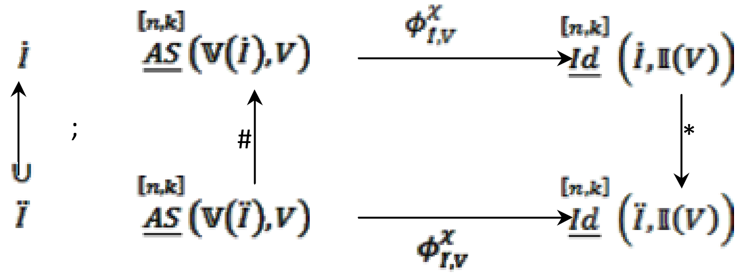
The extended adjunction gives rise to a function ϕ^χ maps every ordered pair $\langle I, V\rangle$ consists of an ideal $I \in k[x_1, \cdots, x_n]$ and affine algebraic set $V \in \mathbb{A}_k^n$ to the following bijection:

$$\phi_{I,V}^\chi : \overset{[n,k]}{\underline{AS}}(\mathbb{V}(I), V) \overset{\sim}{\longrightarrow} \overset{[n,k]}{\underline{Id}}(I, \mathbb{I}(V))$$

which is exactly the generalization of the bijection $\phi_{I,V}$ that couldn’t be reached, previously in §3, by applying Hilbert’s nullstellensatz. As a result we have the following equivalence:

$$\Big((\mathbb{V}(I) \subseteq V) \Leftrightarrow (I \subseteq \mathbb{I}(V))\Big)_{\forall\, I \in \overset{[n,k]}{\underline{Id}},\, V \in \overset{[n,k]}{\underline{AS}}}$$

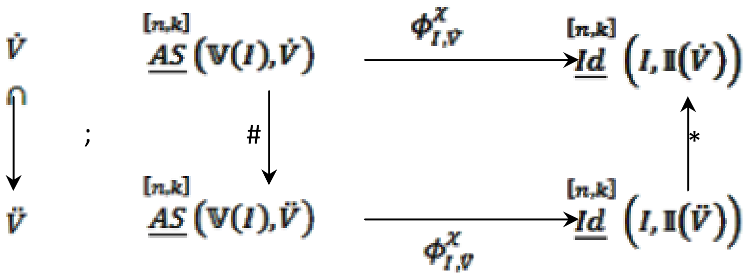
Furthermore, $\phi_{I,V}^\chi$ is natural in both I and V. In other words, for every arrow $I' \hookrightarrow I$ we have the following diagram:



The naturality of ϕ^{χ} in the first component I, makes the previous diagram commutative. Thus, the following holds:

$$\star \circ \phi^{\chi}_{\dot{I},V} \circ \# = \phi^{\chi}_{\ddot{I},V}$$

On the other hand, for every arrow $\ddot{V} \hookrightarrow \dot{V}$, we also have the following diagram:



Again the naturality of ϕ^{χ} in the second component gives the commutativity of the previous diagram:

$$\star \circ \phi^{\chi}_{I,\ddot{V}} \circ \# = \phi^{\chi}_{I,\dot{V}}$$

Therefore, the previous formulae state that ϕ^{χ} as a function with the above equivalence is a natural transformation between the bifunctors:

$$\overset{[n,k]}{\underline{AS}}(\mathbb{V}-,-): \overset{[n,k]}{\underline{Id}} \times \overset{[n,k]}{\underline{AS}} \longrightarrow \underline{Set}, \qquad \overset{[n,k]}{\underline{Id}}(-,\mathbb{I}-): \overset{[n,k]}{\underline{Id}} \times \overset{[n,k]}{\underline{AS}} \longrightarrow \underline{Set}$$

given in the following matter:

$$\phi^{\chi}: \overset{[n,k]}{\underline{AS}}(\mathbb{V}-,-) \dot{\rightarrow} \overset{[n,k]}{\underline{Id}}(-,\mathbb{I}-)$$

4.2 $\quad \varepsilon\mathbb{V} \circ \mathbb{V}\chi = \mathbb{1}_{\mathbb{V}}$:

Using the properties of adjoint functors, we find by taking the following natural transformations:

$$(\mathbb{V}\chi : \mathbb{V} \circ \mathrm{Rad} \dot{\rightarrow} \mathbb{V}) = \left(\mathbb{V}\chi_I : \mathbb{V}(\mathrm{Rad}\, I) \hookrightarrow \mathbb{V}(I)\right)_{\forall I \in \underline{Id}}^{[n,k]}$$

$$(\varepsilon\mathbb{V} : \mathbb{V} \doteq \mathbb{V}) = \left(\varepsilon\mathbb{V}_I : \mathbb{V}(I) = \mathbb{V}(I)\right)_{\forall I \in \underline{Id}}^{[n,k]}$$

that the composition:

$$\mathbb{V} \underset{\varepsilon\mathbb{V}}{\dot{\rightarrow}} \mathbb{V} = \mathbb{V} \circ \mathbb{I} \circ \mathbb{V} = \mathrm{Rad} \circ \mathbb{V} \underset{\mathbb{V}\chi}{\dot{\rightarrow}} \mathbb{V}$$

is the identity natural transformation on the contravariant functor $\mathbb{V}$.

4.3 $\chi\mathbb{I} \circ \mathbb{I}\varepsilon = 1_{\mathbb{I}}$:

Considering the dual of our previous result, we obtain, in the course of the following natural transformations:

$$(\chi\mathbb{I} : \mathrm{Rad} \circ \mathbb{I} \doteq \mathbb{I}) = \left(\chi\mathbb{I}_V : \mathrm{Rad}(\mathbb{I}(V)) = \mathbb{I}(V)\right)_{\forall V \in \underline{AS}}^{[n,k]}$$

$$(\mathbb{I}\varepsilon : \mathbb{I} \doteq \mathbb{I}) = \left(\mathbb{I}\varepsilon_V : \mathbb{I}(V) = \mathbb{I}(V)\right)_{\forall V \in \underline{AS}}^{[n,k]}$$

that the composition:

$$\mathbb{I} \underset{\mathbb{I}\varepsilon}{\dot{\rightarrow}} \mathbb{I} = \mathbb{I} \circ \mathbb{V} \circ \mathbb{I} = \mathrm{Rad} \circ \mathbb{I} \underset{\chi\mathbb{I}}{\dot{\rightarrow}} \mathbb{I}$$

is the identity natural transformation on the contravariant functor $\mathbb{I}$ which is obvious considering that both natural transformations $\mathbb{I}\varepsilon$ and $\chi\mathbb{I}$ are also identities on $\mathbb{I}$.

As an example, consider the special case where $n = 1$ and the base field is $k = \mathbb{C}$. Thus, we have the affine complex line $\mathbb{A}_{\mathbb{C}}^1 \cong \mathbb{C}$ and the polynomial ring in one variable $\mathbb{C}[z]$ which forms a principal ideal ring. Now assuming the ideal $I = \langle z^2 + 1 \rangle$ where $z^2 + 1 = (z - i)(z + i) \in \mathbb{C}[z]$ is clearly reducible and both ideals $\langle z - i \rangle$ and $\langle z + i \rangle$ are radical which means that both components:

$$\langle z^2 + 1 \rangle \hookrightarrow \langle z - i \rangle, \qquad \langle z^2 + 1 \rangle \hookrightarrow \langle z + i \rangle$$

are universal from $\langle z^2 + 1 \rangle$ to the contravariant functor $\mathbb{I}$ which gives the following commutative diagram consisting of two commutative.

Squares for all algebraic sets (finite subsets $V \subseteq \mathbb{C}$) in the affine line:

$$
\begin{array}{ccccc}
\langle z-i\rangle & \overset{[1,\mathbb{C}]}{\underline{AS}}(\mathbb{V}(z-i)=\{i\},V) & \xrightarrow{\phi^{\chi}_{\langle z-i\rangle,V}} & \overset{[1,\mathbb{C}]}{\underline{Id}}(\langle z-i\rangle,\mathbb{I}(V)) & \\
\uparrow & \uparrow \#_i & & \downarrow \star_i & \\
\cup & & & & \\
\langle z^2+1\rangle \;; & \overset{[1,\mathbb{C}]}{\underline{AS}}(\mathbb{V}(z^2+1)=\{-i,i\},V) & \xrightarrow{\phi^{\chi}_{\langle z^2+1\rangle,V}} & \overset{[1,\mathbb{C}]}{\underline{Id}}(\langle z^2+1\rangle,\mathbb{I}(V)) & \\
\cap & & & & \\
\downarrow & \downarrow \#_{-i} & & \uparrow \star_{-i} & \\
\langle z+i\rangle & \overset{[1,\mathbb{C}]}{\underline{AS}}(\mathbb{V}(z+i)=\{-i\},V) & \xrightarrow{\phi^{\chi}_{\langle z+i\rangle,V}} & \overset{[1,\mathbb{C}]}{\underline{Id}}(\langle z+i\rangle,\mathbb{I}(V)) &
\end{array}
$$

which leads to:

$$\star_i \circ \phi^{\chi}_{\langle z-i\rangle,V} \circ \#_i = \phi^{\chi}_{\langle z^2+1\rangle,V} = \star_{-i} \circ \phi^{\chi}_{\langle z+i\rangle,V} \circ \#_{-i}$$

5-Table of Terminology:

$\overset{[n,k]}{\underline{AS}}$:	The preorder category of algebraic sets in an affine $\mathbb{A}^n_k$.
$\overset{[n,k]}{\underline{RI}}$:	The preorder category of radical ideals in a polynomial ring $k[x_1,\cdots,x_n]$.
$\overset{[n,k]}{\underline{Id}}$:	The preorder category of all ideals in a polynomial ring $k[x_1,\cdots,x_n]$.
$\overset{[n,k]}{\underline{Id}} \xrightarrow{\text{Rad}} \overset{[n,k]}{\underline{RI}}$:	The contravariant functor that maps each ideal in $k[x_1,\cdots,x_n]$ to its radical .
$\overset{[n,k]}{\underline{RI}} \xrightarrow{In} \overset{[n,k]}{\underline{Id}}$:	The contravariant full inclusion functor that maps every radical ideal in $k[x_1,\cdots,x_n]$ to itself.
$\overset{[n,k]}{\underline{AS}} \xrightarrow{\mathbb{I}} \overset{[n,k]}{\underline{RI}}$:	The contravariant full functor that maps each algebraic set to its corresponding radical ideal.
$\overset{[n,k]}{\underline{RI}} \xrightarrow{\mathbb{V}} \overset{[n,k]}{\underline{AS}}$:	The contravariant full functor that maps each radical ideal to its corresponding algebraic set.

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