Blowing-ups along exceptional curves

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ABSTRACT

Let X be a 3-dimensional complex manifold(variety) , $C \subseteq X$ an exceptional curve. Let us consider $\mu_1 : X_1 \to X$, $E_1 = \mu_1^{-1}(C)$

We prove that $C_1 \subseteq X_1$ is also an exceptional curve,where C_1 is a negative section corresponding to the exact sequence of I_C / I_C^2 .

And I_C is an O_C -ideal.

Key words: Exceptional divisor, Blowing-ups, Resolution of singularities, Algebraic variety, Analytic variety

النفخ في موازاة منحنيات استثنائية

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الملخص

لـيكن X متنوعـاً (منطويـاً) ثلاثـي الأبعـاد ولـيكن $C \subseteq X$ منحنيـاً اسـتثنائياً، نفـرض $\mu_1 : X_1 \to X$ ، $E_1 = \mu_1^{-1}(C)$ نبرهن هنا أن $C_1 \subseteq X_1$ منحنى استثنائي أيضاً, حيث C_1 مقطع سالب يقابل المتتالية التامة ل I_C / I_C^2 .

الكلمات المفتاحية: القواسم الاستثنائية، النفخ، حلول الشذوذات، متنوع جبري، متنوع تحليلي.

1.Introduction

Let X be a 3-dimensional complex manifold, $C \subseteq X$ a closed compact smooth curve and let $\mu_1 : X_1 \to X$ be the blowing-up of X along C. The exceptional divisor $E_1 = \mu_1^{-1}(C)$ is a ruled surface over C. Here $X_1 = \mu_1^{-1}(X)$

There exists at most one section C_1 of the ruling $E_1 \to C$ with $(C_1)^2_{E1} < 0$. We call this a negative section.

If E_1 has a negative section C_1, then let us consider the blowing-up $\mu_2 : X_2 \to X_1$ along C_1. In this way, we have a sequence of blowing – ups

$(B_k) : X_k \xrightarrow{\mu_k} X_{k-1} \xrightarrow{\mu_{k-1}} ... \to X_1 \xrightarrow{\mu_1} X$, the exceptional ruled surfaces E_i on X_i $(1 \le i \le K)$ and the negative sections C_i

on E_i $(1 \le i \le K)$ such that the μ_j is just the blowing-up of X_{j-1} along C_{j-1} and $E_j = \mu_j^{-1}(C_{j-1})$ for $1 \le j \le K$

The purpose of this paper is to answer the following question;

If C is an exceptional curve, when is C_1 an exceptional Curve ?. In the case $C \cong P^1$ and the normal bundle $N_{c/x} \cong O \oplus (-2)$,(P^1 is the 1-dimensional projective space) M. Reid [6] has proved this and also T. Ando [2] has treated this problem .In [1] and [7]the exceptional locus was used to study properties of simple singularities.

2.Exceptional curves

Let E be a locally free sheaf of rank two on a smooth compact curve C.

2.1. Lemma: (1) If E is a semi-stable vector bundle, then there exist no curves Γ on the ruled surface $P_C(E)$ with $\Gamma^2 < 0$.

(2) If E is unstable, then there exists a unique (up to isomorphisms) exact sequence.

$$0 \to L \to E \to M \to 0,$$

which satisfies the following conditions:

(i) L and M are invertible sheaves on C,

(ii) $\deg_C L > \deg_C M$.

Proof:

(1). Let π be the ruling $P_C(E) \to C$ and O(1) be the tautological line bundle on $P_C(E)$ with $\Gamma^2 < 0$.

(2). Since E is unstable, there exists an exact sequence satisfying (i) and (ii). Assume that there is another sequence.

$$0 \to L' \to E \to M' \to 0,$$

satisfying (i) and (ii). Since deg M' < deg (det E), the homomorphism $L \to E \to M'$ must be zero. Therefore $L' = L$ and $M' \cong M$.

2.2. Definition :

If E is unstable, we call the exact sequence.

$$0 \to L \to E \to M \to 0,$$

satisfying the above conditions (i) and (ii), the characteristic exact sequence of E. Here we also define $d^+(E) := \deg_C L$,

$d^-(E) := \deg_C M$, and $\delta(E) := d^+(E) - d^-(E)$. When E is the conormal bundle $N_{C/X}$ of a curve $C \subseteq X$, we simply denote $d^{\pm}(E)$ and $\delta(E)$:by $d^{\pm}(C)$ and $\delta(C)$ respectively.

2.3. Definition :

A compact smooth curve C in a smooth threefold X is called an exceptional curve, if there exists a proper bimeromorphic morphism f : $X \to Z$ such that f (C) is a point and that f is isomorphic outside C.

We have the following criterion.

2.4. Proposition :

Let $C \subseteq X$ be a compact smooth curve in a smooth threefold. Then C is an exceptional curve if and only if there exists a coherent O_x–ideal J on a neighbourhood of C satisfying the following condition :

(E) : dim (supp (O_x / J)) = 1 , Supp $(O_x / J) \subseteq C$,

and $(J \otimes_{O_x} O_C)$ / torsion) is an ample vector bundle on C.

Proof. First we assume that C is an exceptional curve. Then there exist two effective cartier divisors S_1 and S_2 on a neighbourhood of C such that $(S_1 .C) < 0$, $(S_2.C) < 0$, and dim $(S_1 \cap S_2)=1$.

Let J be the ideal $O_x(-S_1) + O_x(-S_2)$. Then we have

$J \otimes O \cong (O_C \otimes O_x(-S_1)) \oplus (O_C \otimes O_x(-S_2))$.

Thus J satisfies the condition (E).

Next we assume that there is an O_x–ideal J satisfying the condition (E). By considering the primary decomposition of J,

We have an O_x – ideal $J_o \supseteq J$ such that supp (O_X/J_0) = C and

Supp $(J_0/J) \supseteq C$. Hence there is an injection $(J \otimes O_C/\text{torsion})$

$\to (J_0 \otimes O_C/\text{torsion})$, where rank $(J \otimes O_C/\text{torsion})$=rank

$(J \otimes O_C$ / torsion). Therefore J_0 also satisfies the condition (E).

Let $\mu:V \to X$ be the blowing-up by the ideal J_0, i.e.,

V: =$\underline{\text{Projan}}_X (\bigoplus_{d\geq 0} J_0^d)$.

We have an exceptional Cartier divisor.

E: =$\underline{\text{Projan}}_X (\bigoplus_{d\geq 0} J_0^d / J_0^{d+1})$. Let W be a component of E. If μ (W) is a point, then $0_W \otimes O_V(-E)$ is ample, since $O_V(-E)$ is μ-ample. if $\mu(W)$ is not a point, then $\mu(W)$=C and W is also a component of $\underline{\text{proj}}_C(d \geq J_0^d\ 0_C$ /torsion). Since $(J_0 \otimes 0_C/\text{torsion})$ is an ample vector bundle, $0_W \otimes 0_V(-E)$ is also ample. Therefore $0_E(-E)$ is an ample invertible sheaf. Then by the contraction criterion (cf. [3], [4]), we have a morphism $\nu : V \to Z$ such that $\nu(E)$ is a point and ν is an isomorphism outside E. Therefore we have the contraction f: $X \to Z$ of C.

2.5.Lemma:

Let $C \subseteq X$ be an exceptional curve.

(1) If the conormal bundle $N^{\vee}_{C/x} \cong I_C / I_C^2$ is semi-stable, then I_C / I_C^2 is an ample vector bundle.

(2) If I_C / I_C^2 is unstable, then $d^+(C) > 0$,

Proof. Take an ideal J satisfying the condition (E) and the maximal integer k such that $J \subseteq I^k_C$. Then we have an injection

$J / J \cap I_C^{k+1} \to I_C^k / I_C^{k+1} \cong Sy_m^k(I_C / I_C^2)$.

By the condition (E), $J / J \cap I_C^{k+1}$ is an ample vector bundle. Therefore we have proved (1) and (2) .

Let $C \subseteq X$ be an exceptional curve such that I_C/I^2_C is unstable. Let us consider the blowing-up

μ_1: $X_1 \to X$, $E_1 = \mu_1^{-1}(C)$, and the negative section C_1 corresponding to the chracteristic exact sequence of I_C/I^2_C.

2.6. Theorem :

$C_1 \subseteq X_1$ is also an exceptional curve.

Proof: Let $0 \to L \to I_C/I^2_C \to M \to 0$ be the characteristic exact sequence. Assum that I_C/I^2_C is ample. Then from the natural exact sequence.

$$0 \to 0_{C_1} \otimes 0_{X_1}(-E_1) \to I_{C_1} / I^2{}_{C_1} \to 0_{C_1} \otimes 0_{E_1}(-C_1) \to 0,$$

$$\begin{array}{ccc} \cong & & \cong \\ M & & L \otimes M^{-1} \end{array}$$

and the condition deg L> deg M>0, we see that $I_{C_1} / I^2{}_{C_1}$ is also ample. Next assume that I_C / I^2_C is not ample. Then deg $M \le 0$.

Take an 0_X – ideal J satisfying the condition (E) for $C \to X$. Let us consider the 0_{X1} – ideal $J' := \text{Image}\ (\mu^*{}_1 J \to 0_{X1})$.

Since $J \subseteq I_{C1}$, we have $J' \subseteq O_{X_1}(-E_1)$.

Take the maximal integer λ such that

$J' \subseteq O_{X_1}(-\lambda E_1)$ and let $J_1 := J' \otimes O_{X_1}(\lambda E_1) \to O_{X_1}$. we shall prove that the J_1 satisfies the condition (E) for

$C_1 \to X_1$. Since $(J \otimes O_C / \text{torsion})$ is ample on C,

$(J' \otimes O_C / \text{torsion})$ is also ample on C_1 .

Now we have a natural homomorphism.

$J' \otimes 0_{C1} \to 0_{X1}(-LE_1) \otimes 0_{C1} \otimes 0_{C1} \cong M^{\otimes \lambda}$

Since deg $M \le 0$, this homomorphism must be zero. Therefore

$J_1 \subseteq I_{C1}$. On the other hand, ($J_1 \otimes 0_{C1}$/torsion) is ample, since

$$J_1 \otimes 0_{C1} \cong (J' \otimes 0_{C1}) \otimes (0_{X1} (lE_1) \otimes 0_{C1})$$
$$\cong (J' \otimes 0_{C1}) \otimes M^{\otimes(-\lambda)}.$$

Therefore J_1 satisfies the condtion (E).

2.7.proposition .

Let $C \subseteq X$ be an exceptional curve and let J be an O_X–ideal satisfying the condition (E) for $C \subseteq X$.

Then it is impossible to construct an infinite descending filtration $I^{(K)}$ ($k \geq 0$) of the defining ideal I_C which satisfies the following two conditions (a) and (b) :

(a) $I^{(k)}$ is a coherent 0_X – ideal for all $k \geq 0$ and $J \subseteq \bigcap_{k\geq 0} I^{(k)}$

(b) $I^{(K)} / I^{(K+1)}$ is an 0_C – invertible sheaf and not ample for all $K \geq 0$

Proof: By (a) , we can take the maximal integer k such that

$I \subseteq I^{(k)}$. Then we have an injection $J/J \cap I^{(k+1)} \rightarrow I^{(k)}/I^{(k+1)}$ since ($J\, 0_C$ /torsion) is ample, $J/J \cap I^{(k+1)}$ is also ample.

This contradicts (b).

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