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Semiperfect ring which is Extending for simple modules

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ABSTRACT

Any right R-module M is called a CS-module if every submodule of M is essential in a direct summand of M. A ring is said to be CS-ring if R as a right R-module is CS [9]. In this paper we study semiperfect ring in which each simple right R-module is essential in a direct summand of R. We call such ring as a extending for simple R-module. Here we find that for such rings, every simple R-module is weakly-injective if and only if R is weakly-injective if and only if R is self-injective if and only if R is weakly-semisimple. Examples are constructed for which simple R-module is essential in a direct summand.

Key words: Semiperfect ring, CS-module, extending for simple, weakly-injective, weakly-semisimple ring, self injective ring.

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الحلقة نصف التامة التي هي ممدد لمودول بسيط

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الملخص

نقول عن المودول اليمين M فوق الحلقة R إنه CS– مودول إذا كان كل مـودول جزئـي مـن M أساسياً في مجموع مباشر لـ M.

ونقول عن الحلقة R إنها CS– حلقة يمينية إذا كانت R كمودول يميني على نفسها هي CS– مودول (9).

في هذه المقالة ندرس الحلقات نصف التامة والتي من أجلها كل مودول يميني بـسيط هـو مـودول أساسي في مجموع مباشر لـ R.

ندعو هذه الحلقات بأنها ممدد للمودول البسيط فوق R. وهنا نجد أنه من أجل هـذه الحلقـات كـل مودول بسيط فوق R هو أفقي بضعف عندما وفق عندما تكون الحلقة R مودولاً أفقي بضعف وهذا يكافئ أيضاً إن الحلقة R نصف بسيطة بضعف.

وقد تم بناء أمثلة لمودولات بسيطة والتي من أجلها يكون المودول البسيط أساسياً في مجموع مباشر لها.

الكلمات المفتاحية: الحلقة نصف التامة، CS- مـودول، التمديـد للمـودول البـسيط، المودول الأفقي الضعيف، حلقة نصف بسيطة ضعيفة، حلقة أفقية.

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1. INTRODUCTION

Throughout this paper, unless otherwise stated, all rings have unity and all modules are right unital. For any two right R-modules M and N, a submodule S of M is said to be essential in M denoted by $S \subset M$, if for any non-zero submodule L of M, $S \cap L \neq 0$. R is said to be semiperfect if it has a complete set $\{e_i\}_{i=1}^{n}$ of primitive orthogonal idempotents such that each e_iRe_i is a local ring.

Jor Rad(R) will donote the Jacobson radical of R Soc (M) will denote the socle fo M . The injective hull of the right R- module M is denoted by E(M). The notations in this paper are standard and it may be found in [1] and [2].

2. GENERALIZING THE IDEA OF CS-MODULE

Definition 2.1: We say that M is extending for simple module if for each simple submodule S of M there is a direct summand M', of M such that S is essential in M'[5].

Definition 2.2 : R is said to be extending for simple R-module if R as a right R-module is extending for simple R-module.

Definition 2.3 : For any right R-module M, we take a direct decomposition $M= \Sigma\oplus M_i$. For a submodule N_i of M_i, we call $\Sigma \oplus N_i$ a standard submodule of M with respect to this decomposition $\Sigma \oplus M_i$. Thus a standard submodule means a standard submodule with respect to decomposition into indecomposable modules. For any right R-module M, we note that J(M) and Soc(M) are always standard submodules with respect to any decompositions of M.

Definition 2.4 : Let M and N be two right R- modules. We say that M is weakly N-injective if and only if every map $\phi: N \rightarrow E(M)$ from N into the injective hull E(M) of M may be written as a composition $\sigma \circ \hat{\phi}$, where $\hat{\phi}: N \rightarrow M$ and $\sigma : M \rightarrow E(M)$ is a monomorphism. We say that M is weakly-injective if and only if it is weakly N-injective for every finitely generated module N.

Definition 2.5 : A ring R is said to be right weakly-semisimple if every right R-module M is weakly-injective.

Lemma 2.6 : Let S be any simple submodule of M which is essential in M then M is an indecomposable module.

Proof : Let $M= M_1 \oplus M_2$ where M_1 and M_2 are submodules of M. Given that S is essential in M. Therefore $S \cap M_1 \neq 0$ and $S \cap M_2 \neq 0$. Since S is simple implies that $S \subset M_1$ and $S \subset M_2$. This implies that $S \subset M_1 \cap M_2$ which is contradiction.

Lemma 2.7 : If any right R- module M has essential simple submodules S, then Soc (M) = S.

Proof : Let L and S are two simple submodules of M. Since S is essential in M. Therefore $S \cap L \neq 0$. Emplies that $S \subset L$ or $L \subset S$ i.e. $S = L$. Hence Soc(M) = S.

Lemma 2.8 : Let R be a semiperfect ring and let $e_1, e_2, \ldots, e_m$ be a basic set of primitive idempotents for R. If P_R is projective then there exist sets $A_1, A_2, \ldots, A_m$ (unique to within cardinality and possibily empty) such that

$$P \cong (e_1R)^{(A_1)} \oplus (e_2R)^{(A_2)} \oplus \ldots \oplus (e_kR)^{(A_k)}$$

Proof : See [1, Theorem 27.11, Page 306].

Lemma 2.9 : Suppose that $K_1 \subset M_1 \subset M$, $K_2 \subset M_2 \subset M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \underset{e}{\subset} M_1 \oplus M_2$ if and only if $K_1 \underset{e}{\subset} M_1$ and $K_2 \underset{e}{\subset} M_2$.

Proof : See [1, proposition 5.20(2), Page 75].

Proposition 2.10: Let R be any semiperfect ring such that R_R is extending for simple submodule, then

(i) For any projective R-module P, Soc(P) is essential in P.

(ii) If Q is another projective R- module such that Soc (Q) $\cong$ Soc (P) then $Q \cong P$.

Proof : Since R is semiperfect, we may write $R = e_1R \oplus \ldots \oplus e_nR$, where $P = \{e_1R, e_2R \ldots e_kR\}$ $(k \leq n)$ is an irredundant complete set of representatives for the projective indecomposable R- modules. Let $L = \{S_1, S_2, \ldots, S_k\}$ be an irredundant complete set of representatives for the simple R- modules.

Since R_R is extending for simple submodule hence for any simple submodule S_i there exist a direct summand eR of R such that S_i is essential in eR form Lemma 2.6, eR should be indecomposable R-module. Therefore $eR \cong e_jR$ for some $j \in \{1, 2...., k\}$. Thus we can define a function $f : L \to P$ by $f(S_i) = e_jR$, f must be one-one, hence onto. Also by Lemma 2.7, Soc $(e_iR) \cong S_i$ i.e. Soc $(e_iR) = S_i$ is the unique essential submodule of e_iR. Thus Soc (P) is essential in P as proved for indecomposable projective R-module $e_iR = P$.

Let P be an arbitrary projective R- module . Since R is semiperfect there exist sets A_i , i = 1, 2, ..., k such that

$$P \cong (e_1R)^{(A_1)} \oplus (e_2R)^{(A_2)} \oplus \ldots\ldots \oplus (e_kR)^{(A_k)}$$

By Lemma 2.8, since Soc(P) is standard submodule of P. Therefore

$$Soc(P) \cong (Soc(e_1R))^{(A_1)} \oplus (Soc(e_2R))^{(A_2)} \oplus \ldots\ldots \oplus (Soc(e_kR))^{(A_k)}$$

using Lemma 2.9, we get

$$(Soc(e_1R))^{(A_1)} \oplus (Soc(e_2R))^{(A_2)} \oplus \ldots\ldots \oplus (Soc(e_kR))^{(A_k)}$$

$$\subset_e (e_1R)^{(A_1)} \oplus (e_2R)^{(A_2)} \oplus \ldots\ldots \oplus (e_kR)^{(A_k)}$$

i.e. Soc (P) $\subset_e$ P.

(ii) Let $Q = (e_1R)^{(B_1)} \oplus (e_2R)^{(B_2)} \oplus \ldots\ldots \oplus (e_kR)^{(B_k)}$ be any other projective R-module such that $Soc(Q) \cong Soc(P)$.

Therefore $(Soc(e_1R))^{(B_1)} \oplus (Soc(e_2R))^{(B_2)} \oplus \ldots\ldots \oplus (Soc(e_kR))^{(B_k)}$

$$\cong (Soc(e_1R))^{(A_1)} \oplus (Soc(e_2R))^{(A_2)} \oplus \ldots\ldots \oplus (Soc(e_kR))^{(A_k)}$$

and so by the Krull-Schmidt theorem there is a bijection between A_i and B_i for, i = 1, 2, ... k. Therefore

$$(e_1R)^{(B_1)} \oplus (e_2R)^{(B_2)} \oplus \ldots\ldots \oplus (e_kR)^{(B_k)}$$

$$\cong (e_1R)^{(A_1)} \oplus (e_2R)^{(A_2)} \oplus \ldots\ldots \oplus (e_kR)^{(A_k)} \quad \text{i.e. } Q \cong P.$$

Proposition 2.11 : If R is semiperfect and extending for simple right R- module then R is left perfect.

Proof : We shall show that each cyclic R-module has non- zero socle [4, Lemma 9]. For any cyclic R-module xR, if it is contained in e_iR then since e_iR has essential simple submodule S_i. Therefore $S_i \cap xR \neq 0$ Thus $S_i \subset xR$ i.e. Soc (xR) $\neq 0$.

On the other hand if xR contains any e_iR then obviously Soc (e_iR) $\subset$ Soc (xR) i.e. Soc (xR) $\neq 0$.

Theorem 2.12 : Let R be any semiperfect and extending for simple right R-module, then following conditions are equivalent.

(i) Every right simple R-module is weakly-injective.

(ii) R is weakly-injective.

(iii) R is self-injective ring.

(iv) R is weakly-semisimple ring.

(v) Every right R- module is weakly-injective.

Proof : (i) $\Rightarrow$ (ii) Let $R = e_1R \oplus e_2R \oplus .... \oplus e_kR$ ($k \leq n$) as R is semiperfect and $S = \{ S_1, S_2, ..., S_k\}$ be the irrdundant set of simple R-modules. Given that S_i is weakly injective and S_i is essential in e_iR as R is extending. Therefore e_iR is weakly injective. Also finite direct sum of weakly-injective is weakly-injective. Therefore $R = e_1R \oplus e_2R \oplus .... \oplus e_kR$ is weakly-injective.

(ii) $\Rightarrow$ (iii) Suppose R is weakly-injective. By proposition 2.11, R is left perfect. Over left perfect ring R, R is weakly-injective if and only if R is self injective [6, Lemma 2.8].

(iii) ⇒ (iv) Given taht R is self-injective hence it would be weakly-injective. Every direct summand of R is injective and hence every direct summand of R is weakly injective. Therefore R is weakly-semisimple ring [7, Theorem 2.4].

(iv) ⇒ (v) Since R is weakly-semisimple, therefore every right R-module M will be weakly-injective.

(v) ⇒ (i) Obvious.

Example 2.13 :

(1) Let R= $\begin{bmatrix} Z & Q \\ 0 & Q \end{bmatrix}$ is a weakly R-injective. Here simple R-module [0, Q] = $e_{22}R$ is not weakly R-injective i.e. R is not weakly-semisimple ring and R is also not self-injective ring.

(2) For a Boolean ring R, following are equivalents

(i) R is weakly R-injective.

(ii) R is weakly-semisimple ring.

(iii) R is self-injective ring.

Proof : For any Boolean ring, its injective hull E(R) and classical quotient ring Q(R) of R are same i.e. R= E(R)= Q(R).

Example 2.14 : Let S = $\begin{bmatrix} B & A \\ 0 & A \end{bmatrix}$ where A = Q(x_1, x_2, ... x_n) is a field of rational functions in n indeterminates and B = (x_1^2, x_2^2 , ..., x_n^2) is a subfield of A. Let f : A →B defined by f(x_i) = x_i^2, f(a) = a ∀ a ∈ Q, ∀ i = 1, 2, ... n then B is epimorphic image of A [2, Page 338]

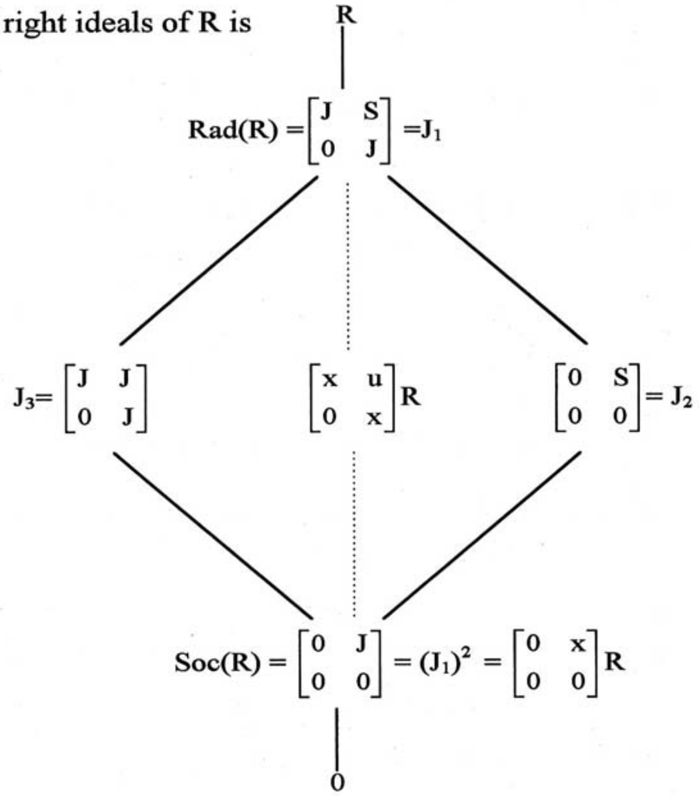
or S = $\frac{Z}{p^2 Z}$

S has three right ideals S, J = Rad(S) = xS and (0).

Also $J^2 \subset J$. Therefore $J^2 = 0$.

Now let $R = \left\{ \begin{bmatrix} a & t \\ 0 & a \end{bmatrix} \mid a, t \in S \right\} \subset \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$ i.e. R is the split extension of the ring S[3].

The lattice of right ideals of R is



where $u \notin J$ in the generator $\begin{bmatrix} x & u \\ 0 & x \end{bmatrix}$ for the cyclic R-module $\begin{bmatrix} x & u \\ 0 & x \end{bmatrix} R$.

Since End $(R_R) \cong R$ is a local ring hence R_R is indecomposable, it is semiperfect. The irredundant complete set of representatives for projective indecomposable R-module contains single element namely R only. And hence irredundant complete set of representatives for simple R-module also contains only single element namely $Soc(R) = \begin{bmatrix} 0 & J \\ 0 & 0 \end{bmatrix}$. Clearly $\frac{R}{Rad(R)} = Soc(R) \subset_e R$. i.e. R is semiperfect and R_R is extending for simple R-module. However the factor ring $\overline{R} = \frac{R}{Soc(R)}$ is also semiperfect but not extending for simple module as $\frac{J_3}{Soc(R)}, \frac{K_u}{Soc(R)}, \frac{J_2}{Soc(R)}$ are three simple $\overline{R}$-modules. Clearly intersection of any two is zero i.e. $\overline{R}$ is not extending for the simple $\overline{R}$-module $\frac{K_u}{Soc(R)}$.

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