On Coflat Module

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ABSTRACT

In this paper we prove that any right R-module M is coflat if and only if (i) $I_M(A \cap B) = I_M(A) + I_M(B)$, for all A and B are finitely generated right ideals of R and (ii) $I_M\, r_R(a) \subseteq Ma$ for all $a \in R$. Examples are constructed which are coflat but not injective R-Module.

Here we find that if R is a right coflat ring, then every finitely generated right ideal is projective if and only if every quotient of right coflat R-module is right coflat if and only if every finitely generated right ideal is projective relative to right coflat R-module.

Key Words: Finitely generated, Projective, Injective, Injective hull, Coflat R-module.

حول المقاس مرافق المسطح

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الملخص

أثبتنا في هذا البحث أن أي مقاس -R يميني M هو مرافق مسطح إذا وفقط إذا:

1) $I_M(A \cap B) = I_M(A) + I_M(B)$ من أجل كل مثالي يميني منتهي التوليد A وB من R.

2) $I_M\, r_R(a) \subseteq Ma$ من أجل كل $a \in R$.

ونظمت أمثلة لمقاسات -R مرافق مسطحة غير تباينية ووجدنا أنه إذا كانت R حلقة مرافق مسطحة فإن كل مثالي يميني منتهي التوليد R هو إسقاطي، إذا وفقط إذا كان كل خارج من مقاس مرافق مسطح يميني هو مرافق مسطح يميني إذا وفقط إذا كان كل مثالي يميني منتهي التوليد إسقاطي بالنسبة لمقاس مرافق مسطح يميني.

الكلمات المفتاحية: منتهي التوليد، إسقاطي، تبايني، غلاف تبايني، مقاس -R مرافق مسطح.

Introduction

Throughout this paper, R is an associative ring with identity. Every module is a unitary right R-module we denote the left annihilator of X in M by $l_M(X)$ and right annihilator of Y in R by $r_R(Y)$ and defined by

$l_M(X) = \{m \in M \mid mx=0, \text{ for all } x \in X\}$ were $X \subseteq R$ and

$r_R(Y) = \{r \in R \mid yr = 0, \text{ for all } y \in Y\}$ where Y is subsect of M or R.

A module M_R satisfies the ψ - Baer criterion in case for every finitely generated right ideal I of R and every R- homomorphism

$$f : I \longrightarrow M$$

there exists an $m \in M$ with $f(x) = mx$, for all $x \in I$.

It has been proved in [2, proposition 1.6] that a module M_R is coflat if and only if it satisfies the ψ - Baer criterion. By direct application of ψ - Baer criterion we have the following result [2, Theorem 1.8].

If $(M_i)_{i\in I}$ be an indexed set of right R- modules. Then

(1). $\prod_{i\in I} M_i$ is coflat if and only if each M_i is coflat.

(2). $\oplus_{i\in I} M_i$ is coflat if and only if each M_i is coflat.

Theorem 1: Any right R – module M_R is coflat if and only if :

(1) $l_M(A \cap B) = l_M(A) + l_M(B)$ for A and B are finitely generated right ideals of R.

(2) $l_M r_R(a) \subseteq Ma$ for all $a \in R$.

Proof : Given that conditions (1) and (2), then to show M_R is coflat.

Let $\varphi : A_R \longrightarrow M_R$ be an R-homomorphism

To prove by induction, take n=1 and $A = a_1R$

For any $x \in r_R(a_1) \Rightarrow a_1 x = 0$

$$\Rightarrow \varphi(a_1 x) = 0$$

$$\Rightarrow \varphi(a_1)\, x = 0$$

and hence $\varphi(a_1) \in l_M[r_R(a_1)] \subseteq Ma_1$ by (2)

therefore $\varphi(a_1) = m\, a_1$ for some $m \in M$

Now take n > 1 and let

$$A = a_1R + a_2R + ... + a_{n-1}R + a_nR$$

We assume (by induction) that there exists $m', m'' \in R$ such that

$$\varphi(a') = m'a' \quad \text{for all } a' \in A' = a_1R + ... + a_{n-1}R$$

and

$$\varphi(a'') = m''a'' \quad \text{for all } a'' \in A'' = a_nR$$

Take any $a \in A' \cap A''$, we have

$$(m'-m'')\, a = 0$$

and hence $m'-m'' \in l_M(A' \cap A'')$

$$= l_M(A') + l_M(A'') \quad \text{by (1)}$$

Therefore we write $m'-m'' = n'-n''$ with suitable elements $n' \in l_M(A')$ and $n'' \in l_M(A'')$.

Let $m = m' - n' = m'' - n''$

if $a = a' + a'' \in A$ with $a' \in A'$ and $a'' \in A''$ then we have

$$\begin{aligned} \varphi(a) &= \varphi(a') + \varphi(a'') \\ &= m'a' + m''a'' \\ &= (m'-n')\,a' + (m''-n'')\,a'' \\ &= m\,a' + m\,a'' \\ &= m\,(a' + a'') \\ &= ma \end{aligned}$$

Hence M_R is coflat.

Conversely, suppose A and B are finitely generated right ideals of R. It is clear

$$l_M(A) + l_M(B) \subseteq l_M(A \cap B) \qquad \text{———————} \qquad (*)$$

Let $x \in l_M(A \cap B)$. We define a mapping

$$\varphi : A + B \longrightarrow M_R$$

given by $\varphi(a + b) = xb$, is a well difined R-homomorphism :

Since M_R is coflat there exists $y \in M$ such that

$$\varphi(a+b) = y\,(a+b) = xb \quad \text{for all } a \in A,\ b \in B$$

In particular, we have

$$0 = \varphi(a) = ya \ \text{ for all } a \in A$$

and hence $y \in l_M(A)$

Further more, we have

$$\begin{aligned} &\varphi(b) = y\,(b) = xb \qquad \text{for all } b \in B \\ &\Rightarrow (x-y)\,b = 0 \\ &\Rightarrow (x-y) \in l_M(B) \end{aligned}$$

If we take $z = x - y$

therefore $x = y + z \in l_M(A) + l_M(B)$

and hence $l_M(A \cap B) \subseteq l_M(A) + l_M(B)$ ——————— (**)

From (*) and (**), we have

$$l_M(A \cap B) = l_M(A) + l_M(B)$$

Now to show condition (2),

Let $y \in l_M(r_R(x))$ then $r_R(x) \leq r_R(y)$

and we may define a map

$$f : xR \longrightarrow yR$$

given by $f(xt) = yt$ for all $t \in R$.

If $xt = xt'$

$\Rightarrow \quad x(t-t') = 0$

$\Rightarrow \quad t-t' \in r_R(x)$

$\Rightarrow \quad y(t-t') = 0$

$\Rightarrow \quad y\,t = y\,t'$

$\Rightarrow \quad f(xt) = f(xt')$

Therefore $f : xR \longrightarrow yR \subset M$ is an R – homomorphism. Since M_R is coflat, then there exists an element $m \in M$ such that

$$yt = f(xt) = mxt \qquad \text{for all } t \in R$$

It following that

$$y = f(x) = mx$$

and hence $y \in Mx$

and consequently $l_M[r_R(x)] \subseteq Mx$

Example 2: Let V be a countably infinite dimensional right vector space over division ring D that is

$$V = \bigoplus_{i=1}^{\infty} x_i D \qquad \text{and } R = \text{End}(V_D)$$

Let $e_i \in R$ be any projection of V onto the "line" x_iD.

Since e_i is a primitive idempotent therefore

$$Re_i \cong V \text{ and } {}_RR = Re_i \oplus R(1-e_i)$$

$$\cong V \oplus R(1-e_i)$$

Let I be any left ideal of R generated by $(e_i)_{i=1}^{\infty}$

then the mapping $f : {}_RI \longrightarrow {}_RV$

defined by $e_i \longrightarrow x_i$ cannot be extended from R to V. That is ${}_RV$ is not injective module which is a direct summand of ${}_RR$. Therefore R as a left R-module is not injective.

R being endomorphism ring of semisimple D-module, is a regular ring. On regular ring every R-module is coflat. Therefore ${}_RR$ is coflat but not injective.

Example 3: Let R be any non-noetherian self-injective ring. Consider the direct sum

$\bigoplus_{i \in I} R_i$, where $R_i \cong R$ foa all $i \in I$. Then being an injective module each R_i is coflat and so

$\bigoplus_{i \in I} R_i$ is coflat as direct sum of coflat modules is coflat. But $\bigoplus_{i \in I} R_i$ is not injective

(since direct sum of injective R-modules is injective, if and only if R is a noetherian ring). As in Example –2.5, R_R is self injective and R is not a noetherian ring.

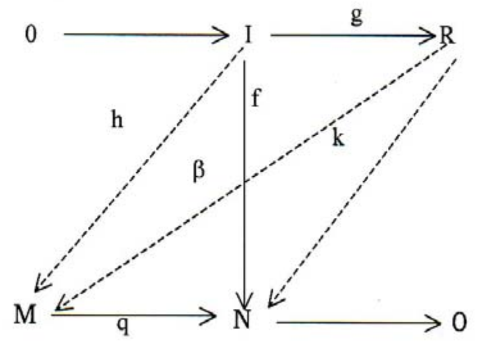
Proposition 4: Following conditions are equivalent for a right coflat ring R :

(1) Every finitely generated right ideal is projective.

(2) Every quotient of right coflat R-module is right coflat.

(3) Every finitely generated right ideal is projective relative to right coflat R-module.

Proff : (1) $\Rightarrow$(2). Let M be any right coflat R-module. Consider the diagram with finitely generated right ideal I of R



Given that I is projective there exists R-homomorphism

$h : I \longrightarrow M$ such that $qoh = f$

Since M_R is coflat R-module, therefore there exists a

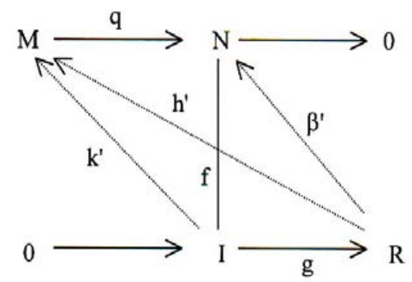
map $\beta : R \longrightarrow M$ such that $h = \beta og$.

then $k = qo\beta : R \longrightarrow N$ gives

$$kog = qo\beta og = qoh = f$$

Hence N is right coflat.

(2) $\Rightarrow$(3); Consider the diagram with right coflat R-module M and I is finitely generated right ideal of R



Given that N is right coflat R-module, therefore there exists a map

$$\beta' : R \longrightarrow N$$

such that $\beta' o g = f$

Since R is projective there exists R-homomorphism

$$h' : R \longrightarrow M$$

such that $q o h' = \beta'$

Then $k' = h' o g : I \longrightarrow M$ gives

$$q o k' = q o h' o g = \beta' o g = f$$

that is I is M-projective

(3) ⇒(1); Given R is a coflat ring and by (3) every finitely generated right ideal I is projective relative to coflat R-module (that is I is R-projective). This implies that I is $F = \oplus R_i$-projective ($R_i \cong R$) that is I-is M-projective for all R-module M_R that is I is projective [8] as any R-module M is a quotient of the free module F.

Proposition 5: Following conditions are equivalent :

(1) Every finitely generated right ideal of R is projective.

(2) Every quotient of right coflat R-module is right coflat.

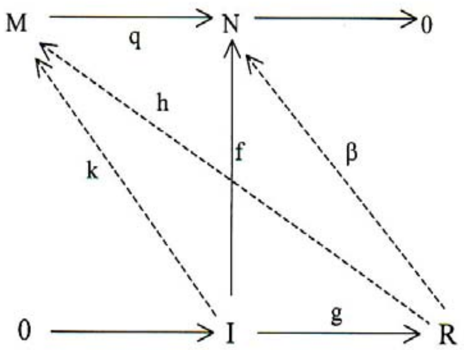
(3) Every quotient of an injective module is right coflat.

Proof : (1) ⇔ (2), From proposition 4.

(2) ⇒(3), It is obvious.

(3) ⇒(1), Let M be any injective R-module. Consider the diagram with

finitely generated right ideal I.



To show I is projective.

Given that N_R is coflat R-module, we have

$$\beta o g = f$$

Since R_R is projective, there exists R-homomorphism

$$h : R \longrightarrow M$$

such that $$q o h = \beta$$

Then $k = h o g : I \longrightarrow M$ gives

$$\begin{aligned} q o k &= q o h o g \\ &= \beta o g \\ &= f \end{aligned}$$

This implies I is M-projective where M is an injective module. Since every module is a submodule of an injective module (injective hull) and $C^p(M)$ is closed under submodule, therefore I is K-projective for all submodule K that is I is projective.

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