On Contact Metric Three-Manifolds Satisfying Certain Curvature Conditions

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ABSTRACT

In this article we study the contact metric three-manifolds M^3 satisfying certain curvature conditions especially $\xi(\lambda)=const.$ In section $3-2$ we find four conditions such that every one is necessary and sufficient for a contact metric three-manifold satisfying

$(*)$ $\quad \xi(\lambda)=A=B=0$, $\lambda^2-1+2a\lambda=0$

to be semi-symmetric. In section $3-4$ we prove that if a contact metric three-manifold M^3 satisfies $(*)$ then either $a=0$ and M^3 is semi-symmetric or $a\neq 0$ and in this case *a* is constant and M^3 cannot be semi-symmetric . In section $3-5$ we prove that if a semi-symmetric contact metric three-manifold satisfies

$\xi(\lambda)=const. \qquad A=const. \qquad B=const.$

(in section 3-6 if $\xi(\lambda)=const.$ $e(\lambda)=const.$ $\varphi e(\lambda)\neq 0.$)

then $a=\xi(\lambda)=A=B=0$

Key Words: Contact 3-Manifolds and Curvature Conditions.

دراسة حول المنطويات المترية التلامسية ثلاثية الأبعاد التي يحقق تقوسها بعض الشروط الخاصة

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الملخص

ندرس في هذا البحث، المنطوي المتري التلامسي ثلاثي الأبعاد M^3 الذي يحقق تنـسور تقوسـه بعض الشروط ولاسيما $\xi(\lambda) = const.$.

نورد بداية أربعة شروط ، كل منها لازم وكافٍ حتى يكون منطو متري تلامسي محقق:

$$(*) \qquad \xi(\lambda) = A = B = 0 \qquad \lambda^2 - 1 + 2a\lambda = 0$$

منطوياً نصف متناظر (أو مثيل المتناظر). نبرهن بعد ذلك أنه إذا حقق منطو متري تلامسي ثلاثـي الأبعاد M^3 العلاقات $(*)$، فعندئذ إما $a = 0$ ، و M^3 سيكون نصف متناظر، أو $a \neq 0$ وعندئـذ a سيكون ثابتاً، ولا يمكن لـ M^3 أن يكون نصف متناظر. وأخيراً نبرهن أنه إذا كانت قـيم المقـادير $\xi(\lambda)$ و A و B ثابتة (وفي حالة أخرى $\xi(\lambda)$ و $e(\lambda)$ ثابتة و $\varphi e(\lambda) \neq 0$) في منطو متري تلامسي ثلاثي الأبعاد ونصف متناظر، فلابد أن يكون: $a = \xi(\lambda) = A = B = 0$

الكلمات المفتاحية: المنطويات التلامسية الثلاثية وشروط التقوس.

1. Introduction

A Riemannian manifold (M, g) is called semi-symmetric if its curvature tensor R satisfies the condition

$$R(X,Y)\cdot R = 0$$

for all vector fields *X, Y* on M, where *R(X,Y)* acts as a derivation on R:

$$\big(R(X,Y)\cdot R\big)(U,V,W) = R(X,Y)\big(R(U,V)\ W\big) -$$
$$- R\big(R(X,Y)U,V\big)W - R\big(U,R(X,Y)V\big)W - R(U,V)\big(R(X,Y)W\big)$$

G. Calvaruso and D. Perrone in [3] have studied the semi-symmetric contact metric three-manifold with Ricci curvature $\rho(\xi,\xi)$ constant, which means $\xi(\lambda)=0$.

In this article we study the contact metric three-manifold M^3 satisfying certain curvature conditions especially $\xi(\lambda) = const.$

2. Preliminaries

A contact metric manifold is a $(2n+1)$-dimensional differentiable manifold M^{2n+1} which carries a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. Every contact metric manifold has an underlying contact Riemannian structure (φ,ξ,η,g), where ξ is a global vector field (called the characteristic vector field or Reeb vector field), φ a global tensor field of type $(1,1)$ and g a Riemannian metric (called associated metric).

These structure tensors satisfy :

$$\eta(\xi)=1,\ \varphi^2 = -I + \eta\otimes\xi,\ \eta(X) = g(\xi,X)$$
$$d\eta(X,Y) = g(X,\varphi Y),\ g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$
$$d\eta(\xi,X)=0$$

We denote by ∇ the Riemannian connection of g, and by R the corresponding Riemannian curvature tensor given by :

$$R(X,Y) = R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X,\nabla_Y]$$

The Ricci tensor of type $(0,2)$, the corresponding Ricci operator and the scalar curvature are respectively indicated by ρ, Q, and r. If r is constant then M^{2n+1} is said to be conformally flat .

In the theory of contact metric manifolds the tensor fields

$h = \frac{1}{2}\mathcal{L}_\xi \varphi$ and $\ell = R(\cdot,\xi)\xi$

where $\mathcal{L}$ is the Lie derivation , play a fundamental role. On every contact metric manifold M^{2n+1} the following relations hold [4]

$$\varphi\xi = h\xi = \ell\,\xi = 0 \qquad \eta \circ \varphi = \eta \circ h = 0$$

$$\nabla_X \xi = -\varphi X - \varphi h X \qquad h\varphi = -\varphi h$$

$$\nabla_\xi h = \varphi - \varphi\ell - \varphi h^2 \qquad \varphi\ell\varphi - \ell = 2(\varphi^2 + h^2)$$

$$tr\, h = tr\, h\varphi = 0 \qquad tr\,\ell = g(Q\xi,\xi) = 2n - tr\, h^2$$

$$\nabla_\xi \varphi = 0 \qquad \nabla_\xi \xi = 0$$

If f is a real function and the almost complex structure J on $M^{2n+1} \times R$ defined by :

$$J(Y, f\frac{d}{dt}) = (\varphi Y - f\xi\ , \eta(Y)\frac{d}{dt})$$

is integrable, then the structure is said to be normal and the manifold is called Sasakian .

Let now $(M^3, \eta, g, \xi, \varphi)$ be a three-dimensional contact metric manifold. Let U be the open subset of M^3 where $h \neq 0$ and V the open subset of points $p \in M^3$ such that $h = 0$ in a neighborhood of p . For any point $m \in U \cup V$ there exist a local orthonormal basis $\{\xi, e, \varphi e\}$ $([2]\, p34)$ of smooth eigen vectors of h in a neighborhood of m . On U we put $he = \lambda e$ $h\varphi e = -\lambda\varphi e$

where λ is a non-vanishing smooth function which is supposed to be positive .

$2-1$ **PROPOSITION** [5]

On U we have

$$\nabla_\xi \xi = 0 \quad \nabla_\xi e = -a\varphi e \qquad \nabla_\xi \varphi e = a e$$

$$(2,1) \qquad \nabla_e \xi = -(\lambda+1)\varphi e \qquad \nabla_e e = -\mu\varphi e$$

$$\nabla_e \varphi e = (\lambda+1)\xi + \mu e$$

$$\nabla_{\varphi e} \xi = -(\lambda-1)e \quad \nabla_{\varphi e} e = (\lambda-1)\xi + \sigma\varphi e$$

$$\nabla_{\varphi e}\varphi e = -\sigma e \qquad \nabla_\xi h = 2ah\varphi + \xi(\lambda)s$$

where a is a smooth function ,

$$\mu = \frac{1}{2\lambda}\{(\varphi e)(\lambda) + A\}\,, \qquad \sigma = -\frac{1}{2\lambda}\{e(\lambda) + B\}$$

$A = \rho(\xi, e)$, $B = \rho(\xi, \varphi e)$ and s is the $(1,1)$-tensor defined by $s\xi = 0$, $se = e$, $s\varphi e = -\varphi e$

$2-2$ **PROPOSITION** [3]

The components of the Ricci operator Q with respect to $\{\xi, e, \varphi e\}$, are given by

$$Q\xi = 2(1-\lambda^2)\xi + Ae + B\varphi e$$

$$Qe = A\xi + (\frac{r}{2} - 1 + \lambda^2 + 2a\lambda)e + \xi(\lambda)\varphi e$$

$$Q\varphi e = B\xi + \xi(\lambda)e + (\frac{r}{2} - 1 + \lambda^2 - 2a\lambda)\varphi e$$

From which it follows :

$$(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi + \{\xi(A) + aB\}e + \{\xi(B) - aA\}\varphi e$$

$$\begin{aligned} (\nabla_e Q)e &= \{e(A) + (\lambda+1)\xi(\lambda) + B\mu\}\xi + \\ &+ \{e(\alpha + 2a\lambda) + 2\mu\xi(\lambda)\}e + \\ &+ \{e\xi(\lambda) + 2a(\varphi e)(\lambda) + (2a - \lambda - 1)A\}\varphi e \end{aligned} \qquad (2,2)$$

$$\begin{aligned} (\nabla_{\varphi e} Q)\varphi e &= \{(\varphi e)(B) + (\lambda - 1)\xi(\lambda) + A\sigma\}\xi + \\ &+ \{(\varphi e)\xi(\lambda) - 2ae(\lambda) + (1 - \lambda - 2a)B\}e + \\ &+ \{(\varphi e)(\alpha - 2a\lambda) + 2\sigma\xi(\lambda)\}\varphi e \end{aligned}$$

where : $\alpha = \frac{r}{2} - 1 + \lambda^2$

$2-3$ **PROPOSITION** [3]

The components of R, with respect to the basis $\{\xi, e, \varphi e\}$ are given by:

$$R(\xi, e)\xi = -(\lambda^2 - 1 - 2a\lambda)e + \xi(\lambda)\varphi e$$

$$R(\xi, \varphi e)\xi = \xi(\lambda)e - (\lambda^2 - 1 + 2a\lambda)\varphi e$$

$$R(e, \varphi e)\xi = -Be + A\varphi e$$

$$R(\xi, e)e = (\lambda^2 - 1 - 2a\lambda)e - B\varphi e$$

$$R(\xi, \varphi e)e = -\xi(\lambda)\xi + A\varphi e$$

(2, 3) $$R(e,\varphi e)e = B\xi + (\frac{r}{2} + 2\lambda^2 - 2)\varphi e$$

$$R(\xi, e)\varphi e = -\xi(\lambda)\xi + B e$$

$$R(\xi,\varphi e)\varphi e = (\lambda^2 - 1 + 2a\lambda)\xi - A e$$

$$R(e,\varphi e)\varphi e = -A\xi - (\frac{r}{2} + 2\lambda^2 - 2)e$$

where :

$R(e_i, e_j)e_k = -R(e_j, e_i)e_k$, $i, j, k = 1,2,3$

$e_1 = \xi$, $e_2 = e$, $e_3 = \varphi e$

$2-4$ **PROPOSITION** [3]

Let (M^3,η,g,φ,ξ) be a non-Sasakian contact metric three-manifold . Then M^3 is semi-symmetric if and only if

$$B(\lambda^2 - 1 + 2a\lambda) = A\xi(\lambda)$$

$$A(\lambda^2 - 1 - 2a\lambda) = B\xi(\lambda)$$

(2 , 4) $$AB + \xi(\lambda)(\frac{r}{2} + 2\lambda^2 - 2) = 0$$

$$B^2 - [\xi(\lambda)]^2 + (\lambda^2 - 1 - 2a\lambda)(\frac{r}{2} + 3\lambda^2 - 3 + 2a\lambda) = 0$$

$$A^2 - [\xi(\lambda)]^2 + (\lambda^2 - 1 + 2a\lambda)(\frac{r}{2} + 3\lambda^2 - 3 - 2a\lambda) = 0$$

3.Some classes of Semi-symmetric contact metric manifolds

$3-1$ **LEMMA**

Let (M^3,η,g,φ,ξ) be a contact metric three-manifold . Then the following formulas hold

$$\rho(\xi,\xi) = 2(1-\lambda^2)$$

$$\rho(e,e) = 2a\lambda - \mu^2 - \sigma^2 - 2a - e(\sigma) - (\varphi e)(\mu) =$$

(3 , 1) $$= \frac{r}{2} + \lambda^2 - 1 + 2a\lambda$$

$$\rho(\varphi e,\varphi e) = -2a\lambda - \mu^2 - \sigma^2 - 2a - e(\sigma) - (\varphi e)(\mu) =$$

$$= \frac{r}{2} + \lambda^2 - 1 - 2a\lambda$$

$$r = 2(1 - \lambda^2 + \gamma)$$

where : $\gamma = -\mu^2 - \sigma^2 - 2a - e(\sigma) - (\varphi e)(\mu)$

<u>Proof :</u>

We have

$$\rho(\xi,\xi) = g\big(R(\xi,\xi)\,\xi,\xi\big) + g\big(R(\xi,e)\,\xi,e\big) + g\big(R(\xi,\varphi e)\ \xi,\varphi e\big)$$

And if we use $(2,3)$ we obtain:

$$\rho(\xi,\xi) = 2(1-\lambda^2)$$

We have also :

$$\rho(e,e) = g\big(R(e,\xi)\,e,\xi\big) + g\big(R(e,e)\,e,e\big) + g\big(R(e,\varphi e)\ e,\varphi e\big)$$

And if we use $(2,1)$ and $(2,3)$ we find :

$$g\big(R(e,\xi)\,e,\xi\,\big) = -(\lambda^2 - 1 - 2a\lambda)$$

$$R(e,\varphi e)\ e = \nabla_{[e,\varphi e]}e - \nabla_e \nabla_{\varphi e} e + \nabla_{\varphi e} \nabla_e e$$

$$= \nabla_{2\xi + \mu e - \sigma\varphi e} e - \nabla_e \big((\lambda - 1)\xi + \sigma\varphi e\big) + \nabla_{\varphi e}(-\mu\varphi e)$$

$$= [-2\lambda\sigma - e(\lambda)]\xi + \big\{\lambda^2 - 1 - \mu^2 - \sigma^2 - 2a - e(\sigma) - (\varphi e)(\mu)\big\}\varphi e$$

$$= B\xi + \big\{\lambda^2 - 1 - \mu^2 - \sigma^2 - 2a - e(\sigma) - (\varphi e)(\mu)\big\}\varphi e$$

And then :

$$\rho(e,e) = 2a\lambda - \mu^2 - \sigma^2 - 2a - e(\sigma) - (\varphi e)(\mu)$$

Similarly we obtain the value of $\rho(\varphi e,\varphi e)$, and hence we have:

$$r = \rho(\xi,\xi) + \rho(e,e) + \rho(\varphi e,\varphi e)$$

$$= 2\big\{(1-\lambda^2) - \mu^2 - \sigma^2 - 2a - e(\sigma) - (\varphi e)(\mu)\big\}$$

Now it follows easily from $(2,4)$ and $(3,1)$:

$3-2$ **COROLLARY**

Let (M^3,η,g,φ,ξ) be a contact metric three-manifold satisfying

$$\xi(\lambda) = A = B = 0,\ \ \lambda^2 - 1 + 2a\lambda = 0$$

$$\lambda^2 - 1 - 2a\lambda \neq 0$$

Then M^3 is semi-symmetric if and only if one of the following conditions is satisfied :

(I) $r = 4(1-\lambda^2)$

(II) $r = 8a\lambda$

(III) $\rho(\varphi e, \varphi e) = 0$

(IV) $\rho(e, e) = 4a\lambda$

$3-3$ **COROLLARY**

Let $(M^3, \eta, g, \varphi, \xi)$ be a contact metric three-manifold satisfying

$R(\xi, \varphi e)\, \xi = 0$, $R(e, \varphi e)\, \xi = 0$, $R(e, \varphi e) \varphi e = 0$

Then M^3 is semi-symmetric .

$3-4$ PROPOSITION

Let $(M^3, \eta, g, \varphi, \xi)$ be a contact metric three-manifold satisfying

$(3-2)$ $\xi(\lambda) = 0$, $A = B = 0$, $\lambda^2 - 1 + 2a\lambda = 0$

Then either $a = 0$ and M^3 is semi-symmetric or $a \neq 0$. In this case a is constant , $r = 4a(\lambda - 1)$ and M^3 cannot be semi-symmetric .

Proof .

Differentiating $\lambda^2 - 1 + 2a\lambda = 0$ with respect to ξ we get $\xi(a) = 0$. Next differentiating with respect to e , we obtain:

$(3-3)$ $\lambda\big(e(\lambda) + e(a)\big) = -a\, e(\lambda)$

We have the well-known formula

$(3-4)$ $\frac{1}{2}X(r) = \sum_{i=1}^{n} g((\nabla_{e_i} Q)e_i, X)$

which holds for any vector field X of n-dimensional Riemannian manifold where $\{e_i\}$ is an arbitrary basis .

If we put $X = e$, $e_1 = \xi$, $e_2 = e$, $e_3 = \varphi e$

and using $(2,2)$ and $(3,2)$ we obtain :

$$\frac{1}{2}e(r) = g\big((\nabla_{\xi} Q)\,\xi, e\big) + g\big((\nabla_{e} Q)\, e, e\big) + g\big((\nabla_{\varphi e} Q)\,\varphi e, e\big)$$

$$= \frac{1}{2}e(r) + 2\lambda[e(\lambda) + e(a)]$$

Since $\lambda \neq 0$, it follows :

$e(\lambda) + e(a) = 0$

Using $(3,3)$ we obtain :

$ae(\lambda)=0$

Then either $a=0$, and from $(3,2)$ we find that $\lambda=1$ and hence all the relations $(2,4)$ are satisfied ,which means that M^3 is semi-symmetric, or $a\neq 0$, $e(\lambda)=0$ and then also $e(a)=0$.

If we apply $(3,4)$ and $(2,2)$ to compute $\frac{1}{2}\varphi e(r)$ we obtain :

$(\varphi e)(\lambda)=(\varphi e)(a)$

Differentiating the last relation of $(2,2)$ with respect to φe and using the last equation we find: $\varphi(e)(a)\cdot(2\lambda+a)=0$, and since $a=\frac{1-\lambda^2}{2\lambda}$, the last formula becomes:

$(\varphi e)(a)\cdot(3\lambda^2+1)=0$

Thus $(\varphi e)(a)=0$ and so $(\varphi e)(\lambda)=0$

Now from

$\xi(a)=e(a)=(\varphi e)(a)=0$

$\xi(\lambda)=e(\lambda)=(\varphi e)(\lambda)=0$

We can conclude that a and λ are constants .

Using $A=B=\mu=\sigma=0$ and from the relations $(3,1)$ we find :

$\rho(\xi,\xi)=2(1-\lambda^2)$

$\rho(e,e)=2a(\lambda-1)$

$\rho(\varphi e,\varphi e)=-2a(\lambda+1)$

$r=4a(\lambda-1)$

Finally if we assume that M^3 is semi-symmetric then according to lemma $3,3$ of [3] we have $a=0$ which contradicts our assumption .

$3-5$ **THEOREM**

Let M^3 be a semi-symmetric contact metric three-manifold satisfying :

(3,5) $\xi(\lambda) = const.$, $A = const.$, $B = const.$

Then $a = \xi(\lambda) = A = B = 0$

and M^3 is either flat or sasakian .

<u>Proof</u>

We multiply the first relation of $(2,4)$ by B , and the second by A , then we subtract the first product from the second product and we get

$$B^2(\lambda^2 - 1 + 2a\lambda) - A^2(\lambda^2 - 1 - 2a\lambda) = 0$$

Using $(2,4)$ to express B^2 and A^2 we find

$$4a\lambda\left\{(\lambda^2 - 1)^2 - 4a^2\lambda^2 - [\xi(\lambda)]^2\right\} = 0 \quad (3,6)$$

Then either $a = 0$ and we shall study the case, or $a \neq 0$ and we shall prove that this case can not occur.

If $a = 0$, then the first, second, fourth and fifth relations of $(2,4)$ become :

$$B(\lambda^2 - 1) = A\xi(\lambda)$$

$$A(\lambda^2 - 1) = B\xi(\lambda)$$

$$B^2 - [\xi(\lambda)]^2 + (\lambda^2 - 1)(\frac{r}{2} + 3\lambda^2 - 3) = 0$$

$$A^2 - [\xi(\lambda)]^2 + (\lambda^2 - 1)(\frac{r}{2} + 3\lambda^2 - 3) = 0$$

From which it follow A= ±B, and the relations (2,4) are equivalent to :

$$A(\lambda^2 - 1) = \pm A\xi(\lambda)$$

$$\pm A^2 + \xi(\lambda)(\frac{r}{2} + 2\lambda^2 - 2) = 0 \quad (3,7)$$

$$A^2 = B^2 = [\xi(\lambda)]^2 + (\lambda^2 - 1)(\frac{r}{2} + 3\lambda^2 - 3)$$

If $A = 0$ then [3]

$$\xi(\lambda) = A = B = a = 0$$

and M^3 either is flat or has constant curvature 1 .

We shall prove that the case $A \neq 0$ can not occur .

In fact consider $U_1 = \{p \in M^3 / A \neq 0 \text{ at } p\}$. Then from the first relation of $(3,7)$ we get on U_1

$$const. = \xi(\lambda) = \pm(\lambda^2 - 1)$$

from which it follows that $\lambda = const.$ and hence $\xi(\lambda) = 0$, and the second relation of $(3,7)$ now yields $A = 0$ on U_1 which can not occur . Therefore , U_1 is empty , and $A = 0$ on M^3.

We assume now $a \neq 0$. Differentiating the third relation of $(2,4)$ with respect to ξ we get

$$\xi(\lambda)\left[\frac{\xi(r)}{2} + 4\lambda\xi(\lambda)\right] = 0$$

If $\xi(\lambda) = 0$ on some neighbourhood U_2 , then $a = 0$ [3] , which contradicts our assumption .

Let $\xi(\lambda) \neq 0$ and $\frac{\xi(r)}{2} + 4\lambda\xi(\lambda) = 0$.

Then:

$$(3,8) \qquad \xi(r) = -8\lambda\xi(\lambda)$$

From $(3,6)$ we find :

$$(3,9) \qquad [\xi(\lambda)]^2 = (\lambda^2 - 1)^2 - 4a^2\lambda^2$$

We add the fourth and fifth relations of $(2,4)$, and we use $(3,9)$ to express $[\xi(\lambda)]^2$.

We obtain :

$$(3,10) \qquad A^2 + B^2 = r(1 - \lambda^2) - 4(1 - \lambda^2)^2$$

Differentiating $(3,10)$ with respect to ξ we find :

$$(1 - \lambda^2)\xi(\lambda) - 2r\lambda\xi(\lambda) - 8(1 - \lambda^2)(-2\lambda)\xi(\lambda) = 0$$

We use $(3,8)$ to express $\xi(r)$.

We get :

$$r = 4(1 - \lambda^2)$$

From $(3,10)$ it follows that: $A^2 + B^2 = 0$

and hence $A = B = 0$ and then $a = 0$ which contradicts our assumption .

$3-6$ **PROPOSITION**

Let M^3 be a semi-symmetric contact metric three-manifold satisfying :

$\xi(\lambda) = const.$, $e(\lambda) = const.$, $\varphi e(\lambda) \neq 0$

Then $a = \xi(\lambda) = A = B = 0$ and M^3 is either flat or Sasakian .

<u>Proof</u>

On M^3 we have the relation $(3,6)$

$$4a\lambda\left\{(\lambda^2 - 1)^2 - 4a^2\lambda^2 - [\xi(\lambda)]^2\right\} = 0$$

If $a = 0$ the conclusion follows as in the proof of theorem $3-5$. We shall prove that the case $a \neq 0$ can not occur .

In fact consider $U_3 = \left\{p \in M^3 / a \neq 0 \text{ at } p\right\}$. On U_3 we have :

(3,11) $$[\xi(\lambda)]^2 = (\lambda^2 - 1)^2 - 4a^2\lambda^2$$

On the other hand , using $(2,1)$ we have:

$$[\xi, e](\lambda) = \xi e(\lambda) - e\xi(\lambda) = (\nabla_\xi e - \nabla_e \xi)(\lambda)$$

Therefore :

$$(\lambda + 1 - a)(\varphi e)(\lambda) = 0$$

and so : $a = \lambda + 1$. then $(3,11)$ becomes

$$[\xi(\lambda)]^2 = (\lambda^2 - 1)^2 - 4\lambda^2(\lambda + 1)^2$$

Differentiating with respect to ξ we obtain :

$$(-12\lambda^3 - 24\lambda^2 - 12\lambda)\xi(\lambda) = 0$$

It is now easy to conclude that $\xi(\lambda) = 0$. In fact if we assume $\xi(\lambda) \neq 0$, then $(-12\lambda^3 - 24\lambda^2 - 12\lambda) = 0$, and differentiating with respect to ξ we get $(-36\lambda^2 - 48\lambda - 12)\xi(\lambda) = 0$ and if we continue in the same way we find at last that $\xi(\lambda) = 0$, which contradicts our assumption . Hence $\xi(\lambda) = 0$, and then $a = 0$ on U_3, which can not occur .

Therefore U_3 is empty and $a = 0$ on M^3 .

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