On the closed subspaces of the inverse limit of finite dimensional vector spaces

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Received 20/09/2005
Accepted 07/03/2006

ABSTRACT

We prove that the sum A + B of closed subspaces A and B of the inverse limit of finite dimensional vector spaces, $V = \varprojlim V_n$ $(n \in N)$ over an algebraically closed field of characteristic 0 is closed.

We extend also the basic fact that every ideal of a finite dimensional semisimple Lie algebra has a unique complement to the case of closed ideals of prosemisimple Lie algebras.

Key Words: Finite Lie Algebras, Inverse limit.

حول الفضاءات الجزئية المغلقة للنهايات المعكوسة للفضاءات الشعاعية (المتجهية) منتهية البعد

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تاريـخ الإيداع 2005/09/20
قبل للنشـر في 2006/03/07

الملخص

أثبت في هذا البحث أن حاصل جمع فضائين جزئيين مغلقين للنهاية المعكوسة لفـضاءات شـعاعية منتهية البعد، هو أيضاً فضاء جزئي مغلق .

كما تم إثبات أن كل مثالي I مغلق للنهاية المعكوسة لجبور لي نصف بسيطة L له فضاء متمم وحيد I' , أي $L = I' \oplus I$

الكلمات المفتاحية: جبور لي المنتهية، النهاية المعكوسة.

1. Introduction

The notion of an inverse sequence and its limit were first discovered in 1929 by Alexandrov. But it was Lefschetz who laid most of the foundations of this subject in his famous book [5] on Algebraic Topology in 1942. Subsequently, inverse limits began to be widely studied and applied in Topology and Algebra such as the inverse limits of linear algebraic groups (usually called the pro-affine algebraic groups) which appear naturally in the representation theory of Lie groups ([4], [6]).

In this paper we prove that if A and *B* are closed subspaces of the inverse limit *V* of finite dimensional vector spaces $V = \varprojlim V_n (n \in \mathbf{N})$, then A + *B* is closed in *V*.

We also extend the basic fact that every ideal of a finite dimensional semisimple Lie algebra has a unique complement to the case of closed ideals of prosemisimple Lie algebras. We prove that if A is a closed ideal of a prosemisimple Lie algebra $L = \varprojlim L_n (n \in \mathbf{N})$ where the L_n are finite dimensional semisimple Lie algebras, then there exists a unique ideal *B of L* such that $L = A \oplus B$. (1)

2. Preliminaries

All Lie algebras in this paper are considered over a fixed algebraically closed field **K** of characteristic 0.

Definition 1. [5, p.31]. Let I be a set with a partial ordering ≤. Suppose I is directed upwards, i.e., for every i, j ∈ I there exists k ∈ I such that i ≤ k and j ≤ k. Let $\{S_i : i \in I\}$ be a family of sets such that for every pair (i, j) ∈ I x I with j ≥ i , there is a map $\pi_{ji} : S_j \to S_i$ satisfying the following two conditions :

(i) π_{ii} is the identity map for every i in I.

(1) 1991 Mathematics Subject Classification. 14L, 16W, 17B45. 1

(ii) If $i \leq j \leq k$, then $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$.

Then $(S_i, \pi_{ji})_{i,j \in I}$ is called an inverse system and the maps $\pi_{ji} : S_j \rightarrow S_i$ are called the transition maps of the inverse system.

The inverse limit of this system, denoted by $\varprojlim S_i$ is the subset of the Cartesian product $\prod_{i \in I} S_i$ consisting of all elements $s = (s_i)_{i \in I}$ such that $\pi_{ji}(s_j) = s_i$ for every $j \geq i$.

A surjective inverse system is an inverse system whose transition maps are surjective.

Definition 2. [1, p.48]. Inverse limit topology: Let $(S_i, \pi_{ji})_{i,j \in I}$ be an inverse system where each set S_i is a topologicol space and each transition map is continuous. Then the inverse limit topology on $\varprojlim S_i$ is the induced topology inherited from the product topology on $\prod_{i \in I} S_i$ which has a basis consisting of the sets of the form $\prod_{i \in I} U_i$ with U_i is an open subset of S_i for every $i \in I$ and $U_i = S_i$ for all but a finitely many $i \in I$. The inverse limit topology is the smallest topology that makes each projection map $\pi_i : \varprojlim S_i \rightarrow S_i$ continuous.

Remark 1. [1, p.49]. Let A be a subset of $\varprojlim S_i$ and let A_i be the image of A under the canonical projection $\pi_i : \varprojlim S_i \rightarrow S_i$. Then the closure of A is given by $\overline{A} = \varprojlim \overline{A_i}$. In particular, if A is closed in $\varprojlim S_i$, then $\varprojlim A_i = A = \overline{A} = \varprojlim \overline{A_i}$.

Moreover, if $A = \varprojlim C_i$ where (C_i) is a closed inverse subsystem of $(S_i, \pi_{ji})_{i,j \in I}$, then A is closed in $\varprojlim S_i$.

Definition 3. **[7, p.600].** Prodiscrete topology: Let $(S_i, \pi_{ji})_{i, j \in I}$ be an inverse system where each set S_i is given the discrete topology, then the resulting inverse limit topology on $\varprojlim S_i$ is called the prodiscrete topology.

Remark 2. [5, p.24]. A topological space X is a T_1- space if each point of every pair of distinct points has a neighborhood which does not contain the other.

Definition 4. Coset topology [2, p. 505]. Let V be a finite dimensional vector space over an arbitrary field **F**. There is a topology on V whose closed sets are the finite unions of sets of the form x + W, where x ranges over V, and W ranges over the subspaces of V. For this topology, V is a compact T_1- space and every linear transformation from V into another such space is continuous and sends closed sets onto closed sets.

In the literature, for instance in [5], we find the definition of an essential element which is the following:

Definition 5. Let $(X_i, \pi_{ji})_{i, j \in I}$ be an inverse system and let $x_i \in X_i$. Then x_i is called an essential element if $x_i \in$ Image (π_{ji}) for all $j \geq I$ in I. That is, $x_i \in \cap_{j \geq i} \pi_{ji}(X_j)$.

However, an essential element may fail to be completed to a full compatible family which is our target; therefore, for convenience, we introduce the following two definitions:

Definition 6. Let $(X_i, \pi_{ji})_{i, j \in I}$ be an inverse system and let $x_i \in X_i$. Then x_i is called super essential element if x_i can be completed to a full compatible family. That is, there exists $x = (x_i)_{i \in I} \in \varprojlim X_i$ where x_i is a component of x.

Definition 7. The inverse system $(X_i, \pi_{ji})_{i, j \in I}$ is called a perfect inverse system if for all $x_i \in X_i$, x_i is a component of a compatible family $(x_i)_{i\in I} \in \varprojlim X_i$, that is, each element $x_i \in X_i$ is a super essential element, so each $\pi_i : \varprojlim X_i \to X_i$ is surjective.

Definition 8. [2, p.504]. An inverse system $(X_i, \pi_{ji})_{i,j \in I}$ of non-empty sets is called a Hochschild-Mostow inverse system, if each X_i can be equipped with a compact T_l topology such that each transition map is a closed continuous map.

Example 1. Finite Sets: Any inverse system of non-empty finite sets is a super inverse system by taking the discrete topology on each given set.

Example 2. Every inverse system of finite dimensional vector spaces (with linear transition maps) is a super inverse system via the coset topology described above.

Theorem 1. *(The projective limit theorem) [2, p. 501]. Let* $(V_i, \pi_{ji})_{i,j \in I}$ *be a Hochschild-Mostow inverse system (i.e. consisting of non-empty compact T_1- spaces V_i and continuous closed maps $\pi_{ji} : Vj \to Vi$). Then:*

(i) $V = \varprojlim V_i \neq 0.$

(ii) If each π_{ji} is surjective, then each canonical projection $\pi_i : V \to V_i$ is surjective.

From the proof of the projective limit theorem given in [2], it follows directly that:

If $(V_i, \pi_{ji})_{i,j \in I}$ is a Hochschild-Mostow inverse system and $v_i \in V_i$ is an essential element, then v_i is super essential.

Theorem 2. *[3, p. 23]. Let L be a finite dimensional semisimple Lie algebra over **K**. Then*

(i) There exist simple ideals L_1, L_2...,L_t [unique up to order] such that
$L = L_1 \oplus \,..\, \oplus L_t$. *(Moreover, each ideal of L is a sum of such simple ideals.*

(ii) All ideals and homomorphic images of L are semisimple.

3. Closed ideals of the inverse limit of finite dimensional vector spaces

Theorem 3. *Let $V = \varprojlim V_i$ be an inverse limit of vector spaces (with linear transition maps). If V is equipped with the prodiscrete topology and V_i is finite dimensional, then the sum A + B, of closed supspaces A and B of V, is closed.*

Proof. Let A_i and B_i be the ith projection of A and B respectively. A and B are closed in the prodiscrete topology, so $A = \varprojlim A_i$, and $B = \varprojlim B_i$.

We shall prove that $A + B = \varprojlim (A_i + B_i)$. Let $a + b \in A + B$, where $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$, so $\pi_{ji}(a_j) = a_i$ and $\pi_{ji}(b_j) = b_i$ for all $j \geq i$. Thus $a + b = (a_i)_{i\in I} + (b_i)_{i\in I}$.

If $\pi_{Kj}(a_k + b_k) = a_j + b_j$ and $\pi_{ji}(a_j + b_j) = a_i + b_i$, where $k \geq j \geq i$, then $\pi_{ki}(a_k + b_k) = \pi_{ki}(a_k) + \pi_{ki}(b_k) = a_i + b_i$. Thus, the $(a_i + b_i)$ are compatible hence, $a + b = (a_i) + (b_i)_{i\in I} = (a_i + b_i)_{i\in I} \in \varprojlim (A_i + B_i)$ and thus $A + B = \varprojlim A_i + \varprojlim B_i \subset \varprojlim (A_i + B_i)$.

On the other hand, let $(a_i + b_i)_{i\in I} \in \varprojlim (A_i + B_i)$. Now, let V_i be equipped with the coset topology whose closed sets C_i are the finite union of sets of the form

$C_i = \bigcup_{k=1}^{n} (x_{ik} + W_{ik})$ where $x_{ik} \in V_i$ and W_{ik} are subspaces of V_i. Since V_i is finite dimensional, then with this topology V_i is a compact T_1- space (for all $i \in I$). Now consider $a_i + (A_i \cap B_i)$, this is closed in V_i (with the coset topology), but V_i is a compact T_1- space, hence $a_i + (A_i \cap B_i)$ is a compact T_1- space too. $(V_i)_{i\in I}$ is an inverse system of finite dimensional vector spaces (with linear transition maps) and we give each V_i its coset topology so (V_i) is a G-super inverse system, i.e. each transition map is a closed continuous map (also left transition maps in each V_i are continuous). Thus $(a_i + (A_i \cap B_i))_{i\in I}$ is a subinverse system consisting of compact

T_1- spaces and continuous closed maps :

$\pi_{ji} : a_j + (A_i \cap B_i) \to a_i + (A_i \cap B_i)$. Then by the projective limit theorem, $L = \varprojlim (a_i + (A_i \cap B_i)) \neq \phi$, i.e. there exists $(a_i + h_i)_{i\in I} \in L$ such that the element $h_i \in A_i \cap B_i$, for every i in I, and for all $j \geq i$, $\pi_{ji}\ (a_j + h_j) = a_i + h_i$, so $\pi_{ji}(a_j) + \pi_{ji}(h_i) = a_i + h_i$. Write $(a_i + b_i)_{i\in I}$ as $(a_i + b_i)_{i\in I} = (a_i + h_i)_{i\in I} + (b_i - h_i)_{i\in I}$, where $a_i + h_i \in A_i$, and $b_i - h_i \in B_i$ for every i in I. The $(a_i + h_i)_{i\in I}$ and the $(b_i - h_i)_{i\in I}$ form compatible inverse subsystems of A and B respectively, thus $(a_i + b_i)_{i\in I} = (a_i + h_i)_{i\in I} + (b_i - h_i)_{i\in I} \in A + B$ and $\varprojlim (A_i + B_i) \subseteq A + B$. Hence, $\varprojlim (A_i + B_i) = A + B$ and $A + B$ is closed in V.

4. Closed ideals of prosemisimple LIE algebras

Theorem 4. *Let L be the inverse limit of a surjective inverse system of finite dimensional semisimple Lie algebras, and let A be a closed ideal of L. Then, there exists a unique ideal B of L such that* $L = A \oplus B$.

Proof. A is closed, thus $A = \varprojlim A_i$ where the A_i is the ith projection of A. Let L_i be the ith projection of L. Since L_i is semisimple by Theorem 2, $L_i = \overset{p}{\underset{t=1}{\oplus}} S_t^i$, p $\in$ **N** where the S_t^i are simple ideals of L_i and since A is an ideal of *L,* A_i is an ideal of L_i. Thus by Theorem 2 also $A_i = \overset{p'}{\underset{t=1}{\oplus}} S_t^i, 1 \le p' \le p$. Similarly $L_j = \overset{m}{\underset{r=1}{\oplus}} S_r^j$, and $A_j = \overset{m'}{\underset{r=1}{\oplus}} S_r^j, 1 \le m' \le m.$ Let $B_i = \overset{p}{\underset{t=p'}{\oplus}} S_t^i$, then clearly $L_i = A_i \oplus B_i$. We note that for a simple ideal S_t^i in A_i there exists a unique S_r^j in A_j such that $\pi_{ji}(S_r^j) = S_t^i$. [This is due to the fact that the preimage of S_t^i is an ideal of L_j, thus it is a sum of some simple S_r^j, If m $\ge$ 2, say for simplicity m= 2 and $\pi_{ji}(S_1^j \oplus S_2^j) = S_t^i$. Then, let π'_{ji} be the restriction of π_{ji} to $S_1^j \oplus S_2^j$. Kernel π'_{ji} which is an ideal of S_1^j, is either 0, which is not the case since π'_{ji} cannot be injective, or $S_1^j \oplus S_2^j$, which is also not the case since $S_t^i \neq 0$, thus the final choice is that Kernel π'_{ji} is either S_1^j or S_2^j. Hence, there exists only one S_r^j such that $\pi'_{ji}(S_r^j) = S_t^i$. This unique S_r^j must be in A_j because $\pi_{ji}(A_j) = A_i$.

Similarly, for a simple ideal $S_{t'}^i$ of B_i there exists a unique simple ideal $S_{r'}^j$ in B_j such that $\pi_{ji}(S_{r'}^j) = S_{t'}^i$. Let π''_{ji} be the restriction of π_{ji} to B_j . If $\pi''_{ji}(B_j) = B_i$, $\pi''_{Kj}(B_K) = B_j$, then $\pi''_{Ki}(B_K) = \pi_{ki}(B_K) = (\pi_{ji} \circ \pi_{Kj})(B_K) = \pi_{ji}(B_j) = B_i$. Thus, $\pi''_{Ki} = (\pi'_{ji} \circ \pi''_{Ki})$. Therefore, the B_i form a surjective subinverse system with transition maps π'_{ji}, the restriction of π_{ji} to B_j. Let B = $\varprojlim B_i$. We claim that L = A $\oplus$ B. Let l $\in$ L, l = (l_i) and l_i can be written uniquely as $a_i + b_i$. Thus l = ($a_i + b_i$). The a_i (respectively the b_i) form a compatible inverse system: This is due to the fact that the l_i are compatible. Let $\hat{\pi}_{ji}$, $\hat{\pi}'_{ji}$ be the restrictions of π_{ji} to a_j and b_j

respectively. If $\hat{\pi}_{ji}(a_j) = a_i$ and $\hat{\pi}_{kj}(a_k) = a_j$, and $\hat{\pi}'_{ji}(b_j) = b_i$ and $\hat{\pi}'_{kj}(b_k) = b_j$, then , since $\pi_{ki}(l_k) = l_i$ we get $\pi_{ki}(l_k) = l_i$ i.e., $\pi_{ki}(a_k + b_k) = a_i + b_i$,

thus $\pi_{ki}(a_k) + \pi_{ki}(b_k) = a_i + b_i$ i.e., $\pi_{ki}(a_k) - a_i = b_i - \pi_{ki}(b_k)$

but $A_i \cap B_i = \{0\}$. Thus $\pi_{ki}(a_k) = a_i$ and $\pi_{ki}(b_k) = b_i$. Hence $\hat{\pi}_{ki}(a_k) = a_i$ and $b_i = \hat{\pi}'_{ki}(b_k)$. Therefore,

$l = (l_i) = (a_i + b_i) = (a_i) + (b_i)$, and consequently L = A $\oplus$ *B.*

Note that given any ideal M_i of L_i, $M_i = (A_i \cap M_i) \oplus (B_i \cap M_i)$. Suppose now that there exists an ideal C $\neq$ 0 of L such that

L = A $\oplus$ *C*. Let C_i be the i th projection of C. Then

$C_i = (A_i \cap C_i) \oplus (B_i \cap C_i)$. But $L_i = A_i \oplus C_i$,

thus $L_i = A_i \oplus (B_i \cap C_i) = A_i \oplus B_i = A_i \oplus C_i$, hence

$B_i \cap C_i = B_i = C_i$, and consequently C = B.

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