I_1-RINGS

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ABSTRACT

The objectiv of this paper is to study the relationship between certain ring R and endomorphism rings of free modules over R. Specifically, the basic problem is to describe ring R, which for it endomorphism ring of all free R-module, is a generalized right Baer ring, right I_1***-ring.***

Call a ring R is a generalized right Baer ring if any right annihilator contains a non-zero idempotent. We call a ring R is right I_1***-ring if the right annihilator of any element of R contains a non-zero idempotent. This text is showing that each right ideal of a ring R contains a projective right ideal if the endomorphism ring of any free R-module is a right*** I_1***-ring. And shown over a ring R, the endomorphism ring of any free R-module is a generalized right Baer ring if and only if endomorphism ring of any free R-module is an*** I_1***-ring.***

Words: *Endomorphism ring, Generalized Bear ring,* I_1*-ring, Annihilator, Projective module.*

الـ I_1 - حـلقـات

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الملخص

إن موضوع هذا البحث هو إلقاء الضوء على بعض خواص الـ I_1 - حلقات مـن اليمـين ووصـف الحلقة التي من أجلها تكون حلقة التشاكلات لأي مودول حر فوقها هي I_1 – حلقة من اليمـين. بالإضـافة إلى ذلك إيجاد الشرط اللازم والكافي كي تكون حلقة التشاكلات لأي مودول حر هي I_1- حلقة من اليمين. سوف نقول عن حلقة ما: إنها I_1- حلقة من اليمين إذا حوى العادم اليميني لأي عنـصر منهـا عنـصراً جامداً مغايراً للصفر، وقد أُثبت ما يأتي: إذا كانت حلقة التشاكلات لأي مودول حر فوق حلقـة مـا، I_1 – حلقة من اليمين فإن كل مثالي يميني في هذه الحلقة يحوي مثالاً يمينياً إسقاطياً. كذلك أُثبـت مـا يـأتي: الشرط اللازم والكافي كي تكون حلقة التشاكلات لأي مودول حر فوق حلقة ما، I_1 - حلقة من اليمين هو أن تكون حلقة التشاكلات لأي مودول حر هي حلقة بيير المعممة من اليمين.

الكلمات المفتاحية: حلقة التشاكلات لمودول، العادم، حلقة بيير المعممـة، المـودول الإسقاطي، الـ I_1 - حلقة.

Introduction

Throughout of this paper R means an associative ring with identity and modules mean unitary right R-modules. For any element $a \in R$ *we denote the right annihilator of* a *by* $r(a) = \{x : x \in R;\ ax = o\}$. *Similarly, the left annihilator of a in R denoted by* $l(a) = \{x : x \in R;\ xa = o\}$.

Call a ring R is a generalized right Baer ring if any right annihilator contains a non-zero idempotent. We call a ring R is right I_1*-ring if the right annihilator of any element of R contains a non-zero idempotent. In this text it is shown that each right ideal of a ring R contains a projective right ideal if the endomorphism ring of any free R-module is a right* I_1*-ring. It is also shown over a ring R, the endomorphism ring of any free R-module is a generalized right Baer ring if and only if endomorphism ring of any free R-module is an* I_1*-ring.*

We begin with the following lemma.

Lemma 1. *For any ring R, the following conditions are equivalent:*

(1)- *For any element* $a \in R$ *there exists an idempotent* $1 \neq f \in R$ *(resp.* $o \neq f \in R$*) such that* $l(a) \subseteq Rf$ *(resp.* $f \in l(a)$*).*

(2)- *For any element* $b \in R$ *there exists an idempotent* $o \neq e \in R$ *(resp.* $1 \neq e \in R$*) such that* $e \in r(b)$ *(resp.* $r(b) \subseteq eR$*).*

(3)- *For any element* $d \in R$ *there exists an idempotent* $1 \neq g \in R$ *such that* $d = dg$ *(resp.* $d = gd$*).*

Proof. $(1) \Rightarrow (2)$. *Let* $b \in R$ *and* $x \in r(b)$, *then* $bx = o$, $b \in l(x)$. *By (1) there exists an idempotent* $1 \neq f \in R$ *such that* $l(x) \subseteq Rf$. *Thus* $b(1-f) = o$, $(1-f) \in r(b)$ *where 1-f is a non-zero idempotent of R.*

$(2) \Rightarrow (3)$. *Let* $d \in R$. *By (2),* $r(d)$ *contains a non-zero idempotent e of R, therefore* $de = o$, $d = d(1-e)$, *where* $1 \neq 1-e \in R$ *is an idmpotent.*

$(3) \Rightarrow (1)$. *Let* $a \in R$, *then* $1 - a \in R$, *by assumption there exists an idempotent* $1 \neq g \in R$ *such that* $(1-a) = (1-a)g$. *Let* $x \in l(a)$, *then* $xa = o$ *and* $x = x(1-a) = x(1-a)g \in Rg$ *for all* $x \in l(a)$, *therefore* $l(a) \subseteq Rf$.

Definition. *We call a ring R is right (resp. left)* I_1*-ring if it satisfies equivalent conditions of lemma 1. A right (resp. left) P.P. ring is a ring in which every principal right (resp. left) ideal is projective [1]. Note that for any element a of a ring R, aR is projective if and only if* $r(a) = eR$, *for some idempotent e of R [2].*

Following [3], a ring R is called a generalized right (resp. left) P.P. ring if for any element a of R, $a^n R$ *(resp.* Ra^n*) is projective for some positive integer* n *(depending on a). It is clear that all right (resp. left) P.P. ring and all generalized right (resp. left) P.P. rings are typical examples of right (resp. left)* I_1*–rings.*

A ring R is called a generalized right Baer ring [4] if any right annihilator contains a non-zero idempotent.

Lemma 2. *If R is a right (resp. left)* I_1*-ring, so is the ring eRe for all idempotent e of R.*

Proof. Let a be an element of eRe, write for $l_e(a)$ *the left annihilator of a in eRe. Note that* $l_e(a) = eRe \cap l(a)$. *Since, R is a right* I_1*-ring, there exists an idempotent* $f \neq 1$ *in R such that* $l(a) \subseteq Rf$. *On the other hand ,* $(1-e) \in l(a)$, *therefore* $(1-e) \in Rf$ *and* $(1-e) = rf = rff = (1-e)f$ *for some* $r \in R$. *So that* $f = ef + (1-e)$ *from which follows* $fe = efe$. *Thus* $g = fe \neq 1$ *is an idempotent in eRe. Suppose* $x \in l_e(a) = l(a) \cap eRe$, *then* $x \in l(a) \subseteq Rf$ *and* $x = xf$, $x = xe = xfe = xg \in eReg$. *Therefore* $l_e(a) \subseteq eReg$.

Lemma 3. *Let F be a free R-module and* $f \in S = End_R(F)$, *then the following conditions are equivalent:*

(1)- *There exists an idempotent* e *of S such that* $e \in r(f)$.

(2)- *Ker f contains a direct summand of F.*

Proof implies immediately from [4, proposition 6].

Lemma 4. *Let F be a free R-module and* $f \in S = End_R(F)$, *then the following conditions are equivalent:*

(1)- *There exists an idempotent e of S such that* $l(f) \subseteq Se$.

(2)- $(\mathrm{Im}\, f)''$ *contains a direct summand of F.*

Proof implies immediately from [4, proposition 4].

For any R-module M,M^* *denotes* Hom_R *(M, R) and S denotes* $End_R(M)$. *Also for any submodule A of M, and for any a non-empty subset I of S, we denote*

$$A' = \{\mu : \mu \in M^*; \mu(A) = 0\}, \quad A'' = \{y : y \in M; A'y = 0\}$$

It is easy to see that $A \subseteq A'', A''$ *is a submodule of M.*

Following [5], let A be a submodule of R-module M, we call a submodule A'' *is a closer to A. And A is called a closed submodule if A=A″.*

Proposition 5. *Let F be a free R-module and* $S = End_R(F)$, *then the following conditions are equivalent:*

(1)- *S is a right* I_I*-ring.*

(2)- *Ker f contains a direct summand of F for any* $f \in S$.

(3)- $(\mathrm{Im}\, f)''$ *contains a direct summand of F for any* $f \in S$.

Proof. $(1) \Leftrightarrow (2)$. *Implies immediately from a definition of a right* I_1*-ring and lemma 3.*

$(1) \Leftrightarrow (3)$. *Implies immediately from a definition and lemma 4.*

Theorem 6. *If* $End_R(F)$ *is a right* I_1*-ring for each free R-module F, then every right ideal of R contains a projective R-module.*

Proof. Let A be a right ideal of R, then we obtain the exact sequence of right R-modules

$$o \to Ker\ f \to F_A \xrightarrow{f} A \to o$$

Where F_A *is a free R-module with basis A. Since* $A \subseteq R$ *then A can be considered as a submodule of F, therefore* $f \in End_R(F)$, *and* $1-f \in End_R(F_A)$. *By hypothesis and proposition 5,* $Ker(1-f)$ *contains a direct summand* F_o *of* F. *Thus* $F = F_o \oplus P$ *where* P *is a submodule of F. Hence* $Ker(1-f) \subseteq Imf \subseteq A$. *Thus* $F_o \subseteq Kerf \subseteq A$. *Since* F_o *is a direct summand of a free* R*-module, then* F_o *is a projective R-module. Thus our proof is completed.*

Lemma 7. *[5, lemma 7]. Let* $F = \sum a_\alpha R, \alpha \in \Omega$ *be a free* R*-module and A be a submodule of F such that* $A \subset M = \sum a_\beta R, \beta \in \Phi \subset \Omega$. *Then* $A'' \subseteq M$.

Theorem 8. *For any ring R the following conditions are equivalent:*

(1)-*End*$_R$*(P) is a generalized right Baer ring for any projective right R-module P.*

(2)- *End*$_R$*(F) is a generalized right Baer ring for any free right R-module F.*

(3)- *End*$_R$*(F) is a right* I_1*- ring for any free right* R*-module* F.

(4)- *End*$_R$*(P) is a right* I_1*- ring for any projective right R-module P.*

Proof. (1)$\Rightarrow$(2). *Is obvious.*

(2)$\Rightarrow$(1). *Let P be a projective R-module, then P is a direct summand of some free R-module F. i.e.* $F = Q \oplus P$. *Since* $End_R(P) \approx eEnd_R(F)e$ *[6] where e the projection of F onto P and e is an idempotent of* $End_R(F)$. *By hypothesis and [4, lemma 2] our proof is completed.*

(2)$\Rightarrow$(3). *It is clear.*

(3)$\Rightarrow$(2). *Let* $F = \sum a_\alpha R \quad , \alpha \in \Omega$ *be a free* R*-module and A be a submodule of F. Suppose* Γ *the set of generators of A. If* $|\Gamma| \leq |\Omega|$, *then A can be considered as Imf for some* $e \in End_R(F)$. *Since* $End_R(F)$ *is a right* I_1*-ring, by proposition 5,* $A'' = (\mathrm{Im}\, f)''$ *contains a direct summand of F. If* $|\Gamma| \phi |\Omega|$ *we put* $F_1 = F + F'$ *where*

$F' = \sum a_\beta R$, $\beta \in \Phi$ *a free R-module and* $|\Gamma| = |\Omega| + |\Phi|$. *Then* $A = Im$ f *for some* $f \in End_R(F_1)$. *Assume,*

$$A'_{F_1} = \{g : g \in F_1^*, g(A) = o\}, \; A'_F = \{g : g \in F^*, g(A) = o\},$$

$$A''_{F_1} = \{y : y \in F_1', A'_{F_1} y = o\}, \; A''_F = \{y : y \in F, A'_F y = o\}$$

Since $End_R(F_1)$ *is a right* I_1*-ring, then, by proposition 5* A''_{F_1} *contains a direct summand Q of* F_1 *.i.e.,* $F_1 = Q \oplus P$, $Q \subseteq A''_{F_1}$. *By lemma 7,* $A''_{F_1} \subseteq F$ *therefore* $F = Q \oplus (P \cap F)$ *i.e.,* A''_{F_1} *contains a direct summand of F. By [4, Theorem 7] our proof completed if* $A''_{F_1} \subseteq A''_F$. *Since any* $f \in Hom_R(F, R)$ *can be considered as an element of* $Hom_R(F_1, R)$ *follows that* $A'_F \subseteq A'_{F_1}$. *But* $A''_{F_1} \subseteq F$ *therefore* $A''_{F_1} \subseteq A''_F$ *i.e.,* A''_F *contains a direct summand of F.*

$(3) \Rightarrow (4)$. *Implies from lemma 2.*

$(4) \Rightarrow (3)$. *Is trivial.*

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