The Upper Fuzzy Index

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ABSTRACT

In this paper we shall study the upper fuzzy index with the upper fuzzy subgroups, also we shall give some new definitions for this subject. On the other hand we shall give the definition of the upper normal fuzzy subgroups, and study the main theorem for this. We shall also give new results on this subject.

Key words: Fuzzy Sets, Upper Fuzzy Subgroups, Upper Normal Fuzzy Subgroups, Upper Fuzzy Cosets**,** Upper Fuzzy Index.

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الملخص

تقدم في هذه الورقة البحثية دراسة عن الدليل المشوش العلوي للزمر الجزئية المشوشة العليا, كمـا تعطي بعض المصطلحات الجديدة في هذا الموضوع فضلاً عن تعريف الزمرة الجزئية المشوشة الناظمية العليا ودراسة النظرية الأساسية عليها, تتضمن هذه الأوراق بعض النتائج المتعلقة بهذا الموضوع أيضاً.

الكلمات المفتاحية: المجموعات المشوشة، الزمر الجزئية العليـا المـشوشة، الزمـر الجزئية العليا الناظمية المشوشة، المرافقات المشوشة العليا، الدليل المشوش العلوي.

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1. Introduction

Zadeh′s classical paper [9] of 1965 introduced the concepts of fuzzy sets and fuzzy set operations. Foster [2] combined the structure of a fuzzy topological space. The study of the fuzzy algebraic structures started with the introduction of concepts of fuzzy subgroups and fuzzy (left, right) ideals in the pioneering paper of Rosenfeld [7]. Anthony and Sherwood [1] redefined fuzzy subgroups using the concept of triangular norm. Several mathematicians [3, 4, 8] followed the Rosenfeld approach in investigating fuzzy algebra where a given ordinary algebraic structure on a given set X is assumed then introducing the fuzzy algebraic structure as a fuzzy subset A of X satisfying some suitable conditions.

In this paper we investigate further the theory of upper fuzzy subgroups and the upper fuzzy index. We also add some result on this subject.

Throughout this paper, G will denote a group and "e" will denote its identity element. Let sup, inf, card, min, max will denote the supermum, infimum, cardinality, minimum, maximum respectively.

To know more of this subject, it is possible to return to Doctorate thesis of Mourad Massa'deh (Damascus University 2008), [5] and [6].

2. Preliminaries

Definition 2.1 [9] let X be a set. A fuzzy set μ on X is just a function

$\mu : X \to [0, 1]$.

Definition 2.2 [7] A fuzzy set μ of a group G is said to be fuzzy subgroups of G if μ satisfies the conditions:

(I) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

(II) $\mu(x^{-1}) = \mu(x)$

For all $x, y \in G$.

Definition 2.3 [8] A fuzzy set μ of a group G is said to be upper fuzzy subgroups of G if μ satisfies the conditions:

(I) $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$

(II) $\mu (x^{-1}) = \mu (x)$

For all $x,y \in G$.

Definition 2.4 [6] Let μ be an upper fuzzy subgroup of a group G. A fuzzy subset satisfying: $[g\mu] (x) = \mu (g^{-1}x)$ for all $x, g \in G$; is called an upper fuzzy left cosets of μ, while that satisfies: $[\mu g] (x) = \mu (xg^{-1})$ for all $x, g \in G$; is called an upper fuzzy right cosets of μ.

In the following study, the upper fuzzy left and right cosets must be fuzzy sets but not necessarily be upper fuzzy subgroups.

Lemma 2.5 If μ is an upper fuzzy subgroup of G, then $\mu (e) \leq \mu (x)$ for every $x \in G$.

Proof:

$$\mu (e) = \mu (x . x^{-1}) \leq \max \{\mu (x), \mu (x^{-1})\}$$
$$= \max \{\mu (x) ,\mu (x)\} = \mu (x).$$

Proposition 2.6

Let G be a group and μ be an upper fuzzy subgroup of G. Then for any $x, y \in G$ such that $\mu (x) \neq \mu (y)$, we have $\mu (xy) = \max \{ \mu (x), \mu (y)\}$

Proof :

Assume that $\mu (y) > \mu (x)$, then :

$\mu (y) = \mu (x^{-1}xy) \leq \max \{ \mu (x^{-1}) ,\mu (xy) \}= \max \{ \mu (x) , \mu (xy) \} = \mu (xy)$

Also μ is an upper fuzzy subgroup, then :

$\mu (xy) \leq \max \{ \mu (x) , \mu (y) \} = \mu (y)$. Thus $\mu (y) \leq \mu (xy) \leq \mu (y)$, this implies that $\mu (xy) = \max \{ \mu (x) , \mu (y) \}$

The same way if $\mu (x) > \mu (y)$. □

Definition 2.7 An upper fuzzy subgroup μ of a group G is called upper normal fuzzy subgroup if $\mu (x y x^{-1}) \leq \mu (y)$, for all $x , y \in G$.

Proposition 2.8 The following conditions are equivalent:

I) G is an upper normal fuzzy subgroup

II) $\mu (x yx^{-1}) = \mu (y)$ for all $x,y \in G$

III) $\mu (xy) = \mu (yx)$ for all $x,y \in G$

Proof :

$(I) \rightarrow (II)$:

$\mu(y) \geq \mu(xyx^{-1})$ for all x , y $\in$ G, on the other hand we need to show $\mu(xyx^{-1}) \geq \mu(y)$ for all x ,y $\in$ G. $\mu(y) = \mu(x^{-1}xyx^{-1}x) \leq \mu(xyx^{-1})$. Then $\mu(y) \leq \mu(xyx^{-1})$, therefore $\mu(xyx^{-1}) = \mu(y)$ for all x , y $\in$ G.

$(II) \rightarrow (III)$:

Since $xy = x(yx)x^{-1}$, then $\mu(xy) = \mu(x(yx)x^{-1}) = \mu(yx)$, then $\mu(xy) = \mu(yx)$ for all x ,y $\in$ G.

$(III) \rightarrow (I)$:

Since $\mu(xyx^{-1}) = \mu((xy)x^{-1}) = \mu(x^{-1}(xy)) = \mu(x^{-1}xy) = \mu(y)$, then $\mu(xyx^{-1}) \leq \mu(y)$, for all x ,y $\in$ G

Thus the conditions are equivalent.□

Definition 2.9 Let μ be an upper fuzzy subgroup of a group G. Then the upper fuzzy index of μ in G is defined by:

$$[G:\mu] = \text{Card}\left\{\begin{array}{l} \textit{the set of the upper fuzzy left cosets of } \mu \textit{ in } G, \\ \textit{that are themseleves are upper fuzzy subgroups} \end{array}\right\}$$

We shall call this type of the upper fuzzy index by general upper fuzzy index, because this type is not restricted by any conditions.

3. Results

Proposition 3.1 If μ is an upper fuzzy subgroup of G , then the value of the upper fuzzy index is equal to the number of elements in μ which is equal to $\mu(e)$ adding for it one , i.e.,

$[G : \mu]$ = card (element of μ which is equal to $\mu(e)$) + 1.

Proof :

For all elements of μ which is the value of it's elements equal to $\mu(e)$.

The upper fuzzy left cosets of its elements are upper fuzzy subgroups; also the upper fuzzy left coset of $\mu(e)$ is also upper fuzzy subgroup. And these satisfy the definition of the upper fuzzy index. Therefore:

$[G : \mu]$ = card (element of μ which is equal to $\mu(e)$) + 1.

Theorem 3.2 If μ is an upper fuzzy subgroup of a group G, and $[G : \mu] = 3$, then μ is an upper normal fuzzy subgroup of G.

Proof :

Suppose that μ is an upper fuzzy subgroup of G and $[G : \mu] = 3$, this means that μ contains two elements having the same value of $\mu(e)$ and let these two elements $\mu(x)$, $\mu(y)$ such that $x, y \in G$, we need to prove that μ is an upper normal fuzzy subgroup, which means that for all $z, h \in G$, we need to show $\mu(zhz^{-1}) \leq \mu(h)$ or $\mu(hz) = \mu(zh)$ or $\mu(hzh^{-1}) \leq \mu(z)$ there exist many cases for $\mu(h)$, $\mu(z)$:

(I) If $\mu(h) = \mu(z)$ and any of these is not equal to $\mu(e)$, then

$$\begin{aligned}\mu(zhz^{-1}) &\leq \max\{\mu(zh), \mu(z^{-1})\}\\ &= \max\{\mu(zh), \mu(z)\}\\ &\leq \max\{\max\{\mu(z), \mu(h)\}, \mu(z)\}\\ &= \max\{\max\{\mu(h), \mu(h)\}, \mu(h)\}\\ &= \mu(h)\end{aligned}$$

Then $\mu(zhz^{-1}) \leq \mu(h)$.

(II) If $\mu(h) \neq \mu(z)$ and any of these is not equal to $\mu(e)$ by proposition 2.6.

$\mu(hz) = \max\{\mu(h), \mu(z)\}$ and

$\mu(zh) = \max\{\mu(z), \mu(h)\}$ thus

$\mu(zh) = \max\{\mu(z), \mu(h)\} = \max\{\mu(h), \mu(z)\} =$

$\mu(hz)$, then $\mu(zh) = \mu(hz)$.

(III) In this case we have two subcases :

Subcase (a). In the case if the element h of G is the same of x or y or e it follows that $\mu(h) = \mu(e)$ and the element z of G is different from x, y and e then $\mu(e) \lneqq \mu(z)$ because there are only two elements in G and their values in μ equal to $\mu(e)$, thus:

$$\begin{aligned}\mu(hzh^{-1}) &\leq \max\{\mu(hz), \mu(h^{-1})\}\\ &= \max\{\mu(hz), \mu(h)\}\\ &\leq \max\{\max\{\mu(h), \mu(z), \mu(h)\}\\ &= \max\{\mu(z), \mu(h)\}\\ &= \mu(z)\end{aligned}$$

Then $\mu(h z h^{-1}) \leq \mu(z)$.

Subcase (b). In the case if the element z of G is the same of x or y or e it follows that $\mu(z) = \mu(e)$ and the element h of G is different from x, y and e then $\mu(e) \gneqq \mu(h)$ because there are only two elements in G and their values in μ equal to μ (e), thus:

$$\begin{aligned} \mu(z h z^{-1}) &\leq \max\{\mu(z h), \mu(z^{-1})\} \\ &= \max\{\mu(z h), \mu(z)\} \\ &\leq \max\{\max\{\mu(z), \mu(h), \mu(z)\} \\ &= \max\{\mu(h), \mu(z)\} \\ &= \mu(h) \end{aligned}$$

Then $\mu(z h z^{-1}) \leq \mu(h)$.

(iv) if the element h of G is the same of x or y or e it follows that

$\mu(h) = \mu(e)$, and the element z of G is the same of x or y or e it follows that $\mu(z) = \mu(e)$ then:

$$\begin{aligned} \mu(z h z^{-1}) &\leq \max\{\mu(z h), \mu(z^{-1})\} \\ &= \max\{\mu(z h), \mu(z)\} \\ &\leq \max\{\max\{\mu(z), \mu(h)\}, \mu(z)\} \\ &= \max\{\max\{\mu(h), \mu(h)\}, \mu(h)\} \\ &= \mu(h) \end{aligned}$$

Then $\mu(z h z^{-1}) \leq \mu(h)$.

From the above, all cases that we have studied, we conclude that μ is upper normal fuzzy subgroup of G.

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