Basic (nxm) real Matrix operations by using complex numbers

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ABSTRACT

In this paper, we generalize the study of the mathematical operations over (2x2), (3x3) and (4x4) real matrices by using the complex numbers which was introduced by (Ide. 1990,1993,1996), for any real matrices (nxm).

This method of arithmetic operations of matrices by using the complex numbers and its properties is simple, easy and fast to program on computer.

The importance of this method appears in the applications of problems which use the arithmetic operations of matrices, especially in physics.

Key Words: Real Matrix Operations, Product, Columns, Rows, Quaternion, Inverse**.**

العمليات الرياضية على المصفوفات الحقيقية (nxm) باستخدام الأعداد العقدية

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الملخص

في هذا العمل تم تعميم العمليات الرياضية على المصفوفات الحقيقيـة (2x2) و(3x3) و(4x4) التـي كانت قد درست من قبل (Ide.N, 1990,1993, 1996) وذلك على المصفوفات من المرتبـة (nxm) بغيـة تبسيط تلك العمليات وبرمجتها على الحاسوب، ثم استخدامها في التطبيقات العمليـة مـستخدمين لـذلك أقواساً تضم أعداداً عقدية كتبت بعلاقات رياضية بينها وبين عناصر المصفوفات الحقيقية (إيزومـورفيزم بين هاتين الكتابتين)، وتم إجراء العمليات الرياضية على أعداد تلك الأقواس (العقدية) بدلاً من إجرائهـا على المصفوفات الحقيقية. كما تم ضبط عدد العمليات الرياضية وخاصة عملية الضرب باعتبارها الأكثـر كلفة للحاسوب. أخيراً تمت معالجة بعض الأمثلة العددية لإيجاد مقلوب المصفوفة (باعتبار هـذه العمليـة هي الأكثر تعقيداً على المصفوفات) بهذه الطريقة العقدية، وتمت مقارنة النتيجة التي حصلنا عليهـا بعـد برمجتها على الحاسوب بلغة الفورتران مع النتيجة الكلاسيكية لمقلوب المصفوفات ووجد أن الخطأ بـين النتيجتين كان شبه معدوم (جزء من مليون وأحياناً أقل) ومن ثم فإن هذه الطريقة يمكن الاعتماد عليهـا في الحسابات وخاصة في المسائل الفيزيائية حيث يلاحظ في كثير من الحالات تطابق الكثير من الأعـداد المركبة لهذه المصفوفات الحقيقية؛ مما يخفض عدد العمليات الرياضية.

الكلمات المفتاحية: العمليات على المصفوفات الحقيقية، الجداء، الأعمـدة، الأسـطر، الرباعي، المقلوب.

1-Introduction :

Here we generalize the mathematical operations over the real matrices 2x2, 3x3, and 4x4 which were studied by (Ide.N 1990, 1993, 1996) using the complex numbers, for any real matrix (nxm). The real matrices will be written in the form of brackets of complex numbers. For this purpose, any (2x2) real matrix can be written by two brackets of two complex numbers A and B, (Ide, 1990), in the form :

$$M=\{A,B\}. \qquad \textbf{(1)}$$

where A and B are given as a linear combination of elements of matrix M.

Also, any M(3x3)real matrix can be written by two brackets of the form of complex numbers A,B,C,L and a scalar S in the form :

$$M=\{A,B,C,L,S\} \qquad \textbf{(2)}$$

where C and L are (2x1) column vector and (1x2) row vector respectively. In the begining of this study, we distinguish between the (nxn) square matrices and the (nxm) rectangular matrices.

Finally, we shall study the classical matricial operations using these brackets of complex numbers and estimate the number of product operations needed for multiplying two matrices, since this operation is very costly with respect to computer time.

2- The product :

We distinguish between two cases: the (nxn) square matrices and the (nxm) rectangular matrices.

2-1- The product for the (nxn) square matrices :

We shall study this case for an even n and then for an odd n.

<u>2-1-1-</u> <u>For an even n</u>:

Let M (n,n) be a real square matrix, where n is an even number. Then, by dividing this matrix into (2x2) square matrices we obtain $\frac{n\,.\,k}{2}$ matrices.

We know that any (2x2) real matrix can be written by the form (1), then, we can write matrix M in the form :

$$M=[\{A_1,B_1\},\{A_2,B_2\},\ldots\ldots\ldots\ldots,\{A_n,B_n\}] \qquad \textbf{(3)}$$

where, $\{A_1, B_1\}, \{A_2, B_2\}$, ..., $\{A_n, B_n\}$ are the brackets of the divided matrix M taken successively row by row from left to right.

For multiplying matrix M'(n,n) by matrix M"(n,n), we write them by the brackets form as :

$$M' = [\{A'_1, B'_1\}, \{A'_2, B'_2\}, \ldots\ldots\ldots, \{A'_n, B'_n\}] \quad \textbf{(4)}$$

$$M'' = [\{A''_1, B''_1\}, \{A''_2, B''_2\}, \ldots\ldots\ldots, \{A''_n, B''_n\}] \quad \textbf{(5)}$$

Then, the product of these matrices can be performed as a multiplication of their dividing matrices which is written by the brackets form. i, e;

$$M = M' \,.\, M'' = [\{A_1, B_1\}, \{A_2, B_2\}, \ldots\ldots\ldots, \{A_n, B_n\}] \quad \textbf{(6)}$$

such that, any of these brackets $\{A_i, B_i\}$ is obtained as the sum of n products of the brackets of M' and M".

2-1-2-Estimation of the number of product operations :

Calculating the number of product operations for this multiplication, we find that we need $\frac{n^3}{2}$ products of complex numbers where as in classical matricial product we need $\boldsymbol{n}^3$ products of real numbers.

2-1-3-Examples :

1)- For n = 4, in this case we have four (2x2) real matrices, matrix M will be written in the form :

$$M = [\{A_1, B_1\}, \{A_2, B_2\}, \{A_3, B_3\} \{A_4, B_4\}] \quad \textbf{(7)}$$

and we need 32 products of complex numbers.

2)- For n = 6, we need 108 products of complex numbers.

2-1-4-For n odd :

Let M(nxn)=($\boldsymbol{m_{ij}}$),i, j = 1, 2, ..., n. be a real matrix where n is an odd number, then we divide this matrix into (2x2) real matrices. In this case, it will rest in the last column: $\frac{n-1}{2}$ column vectors of the form:

$$C_1 = [m_{1,n} \quad m_{2,n}]^T, \ldots\ldots, \ldots\ldots C_{\frac{n-1}{2}} = [m_{n-2,n} m_{n-1,n}]^T \qquad (8)$$

and rest in the last row $\frac{n-1}{2}$ row vectors of the form :

$$L_1 = [m_{n,1} \quad m_{n,2}], \ldots\ldots, \ldots\ldots, L_{\frac{n-1}{2}} = [m_{n,1} m_{n,2}] \qquad (9)$$

and one real scalar: S = m_{nn} ; and hence, we can write this matrix by the brackets of complex numbers :

M=[{ A_1, B_1 },..,{ $A_{(\frac{n-1}{2})^2}, B_{(\frac{n-1}{2})^2}$ }, C_1 ,..., $C_{\frac{n-1}{2}}, L_1$,..., $L_{\frac{n-1}{2}}$, S] **(10)**

where C_i and L_i are complex numbers (Ide, 1990).

Now, let M'(nxn) and M"(nxn) be two real matrices where n is an odd number, then we can write M' and M" by the brackets form :

M'=[{ A'_1, B'_1 **},..., {** $A'_{(\frac{n-1}{2})^2}, B©_{(\frac{n-1}{2})^2}$ **},** C'_1 **, ...,** $C'_{\frac{n-1}{2}}, L'_1$ **,**

..., $L'_{\frac{n-1}{2}}$**, S']** **(11)**

and :

M"=[{ A''_1, B''_1 **}, ...,{** $A''_{(\frac{n-1}{2})^2}, B''_{(\frac{n-1}{2})^2}$ **},** C''_1 **, ...,** $C''_{\frac{n-1}{2}}$ **,**

L''_1 **, ...,** $L''_{\frac{n-1}{2}}$ **, S"]** **(12)**

The product of M' and M" by using the brackets of these matrices shall be found, and the following operations over these brackets are defined as follows :

1)- {A , B}. {C , D} = { A.C + B . $\overline{D}$, A.D + B . $\overline{C}$ } **(13)**

2)- The product of a real scalar S by a complex number C, i.e; C.S, is defined as known.

3)- The product of brackets {A, B} by the complex number C, is defined by :

$$\{A, B\} . C = A.C + B . \overline{C} \qquad \textbf{(14)}$$

4) The product of the complex number L by the brackets {A, B} is defined by:

$$L . \{A, B\} = L. \overline{A} + \overline{L} .B \qquad \textbf{(15)}$$

5) The product of the complex number L by the complex number C is defined by :

$$L . C = \overline{L} . C + L . \overline{C} \qquad \textbf{(16)}$$

6) The product of the complex number C and the complex number L is defined by :

$$C . L = \{ C. \overline{L} , C.L \} \qquad \textbf{(17)}$$

where C is a (2x1) vector and L is a (1x2) vector.

After these definitions, we see that the product is completely well defined over these matrices. In fact, these definitions are the same as the classical matricial product, but by using the brackets of complex numbers.

2-1-5- Estimation of the number of product operations :

Calculating the number of product operations needed for multiplying the two matrices M'(n,n) and M"(n,n) by using the brackets of complex numbers, we find that :

$$N_1 = \frac{n^3 + 1}{2} \qquad \textbf{(18)}$$

Product operations of complex numbers are needed which contain one real product operation that appears during the calculation of S.

2-1-6- Examples :

1) For n = 3 :

We have one (2x2) matrix only and two vectors C_1 , L_1 and one real scalar S, in this case N_1 = 14. In classical product we have 27 product operations, (Ide,1990).

2) For n = 5, we have N_1 = 63.

3) For n = 7 we have N_1 = 172.

3- The product for the general case (for the (nxm) rectangular matrices) :

Let M'(n,k) and M"(k,m) be two real rectangular matrices, then for calculating their product, we write: M (n,m) = M'(n,k). M"(k,m) and by using the brackets of complex numbers, we divide them also into (2x2) real matrix, C(1x2) real vector and into L(2x1) real vector. By using the definitions given by relations (13) to (17), then the product of these two matrices can be found by using the complex numbers.

3-1- Estimation of the number of product operations :

For calculating the number of product operations, we distinguish between the following cases of the matrices M'(n,k) and M"(k,m).

3-2- n,k and m are even (number of rows and colomns are even) :

If the numbers n, k and m are even, then the multiplication is of the form: {A, B}. {C, D} only. and we have in this case ($L\frac{n-1}{2}+\frac{n.k}{2}$), (2x2) matrices and the number of product operations (of complex number) is:

$$N_2 = \frac{1}{2}k.n.m \qquad \textbf{(19)}$$

In classical matricial product we need (k.n.m) products of real numbers.

3-2-1-Examples :

1)- We can consider the case (3.1) as a special case i.e; for n is even we take here k=m=n, we find the same number of product in case (3.1).

2)-For M'(2,4) and M"(4,6) we need 24 product operations of complex numbers.

3-3- For the number of rows or columns or all of them are odd :

If the number of rows or columns or all of them are odd, then we divide the two matrices into matrices of the form : {A , B} and into vectors of the forms : $C = [.\ \ .]^T$, L = [. .]

For calculating the number of product operations, we shall distinguish between the following cases of matrices M'(n,k) and M"(k,m):

a)- n odd, k even and m odd
b)- n odd, k even and m even
c)- n even, k odd and m even
d)- n even, k odd and m odd
e)- n odd, k odd and m even
f)- n odd, k odd and m odd
g)- n even, k even and m odd

Examining the two cases (a) and (b) we find that:

$$N_3 = \frac{1}{2} . k . n . m. \qquad \textbf{(20)}$$

3-3-1-Examples :

1)- For n = 5, k = 4 and m = 5, we need 50 product of complex numbers.

2)- For n = 5, k = 4, m = 6. We have : N_3 = 60.

Finally, on generalization of these cases, we find that the number of product operations in classical matricial product is twice the number of product operations using complex brackets.

3-4- Remark :

For the product of vectors, which we divide them into (2x1) and (1x2) vectors, we can consider them as a special cases of matrices and we obtain their product as in case (3.2).

4-The inverse problem :

We can calculate the inverse of a real square matrix by using the brackets of complex numbers i.e; if

M= [{ A_1, B_1},{ A_2, B_2}, ..., { A_n, B_n }] be a (nxn) matrix, then, for finding the inverse :

$$M^{-1} = [\{ A'_1, B'_1 \},\{ A'_2, B'_2 \}, ..., \{ A'_n, B'_n \}] \qquad \textbf{(21)}$$

we must solve the equations :

$$M . M^{-1} = M^{-1} . M = I_n \qquad \textbf{(22)}$$

4-1- Examples :

1- For M(3,3) i.e; for n=3, we get : M^{-1} = {A’, B’, C’, L’, S’ } of the matrix : M = {A, B, C, L, S} where :

A' = $\frac{1}{\det M}$**(S** $\overline{A}$ **- L** $\overline{C}$**);B' =** $\frac{1}{\det M}$**(L C - S B); C' =** $\frac{1}{\det M}$ **(**$\overline{C}$ **B - C** $\overline{A}$**) ;L' =** $\frac{1}{\det M}$**(B** $\overline{L}$**- A L) S' =** $\frac{\det T}{\det M}$ **(23)**

where : det M $\neq 0$, T=(m_{ij}) , i,j = 1,2 and :

det **M = L (C** $\overline{B}$ **-** $\overline{C}$ **A) +** $\overline{L}$ **(**$\overline{C}$ **B - C** $\overline{A}$**) + S (A** $\overline{A}$ **- B** $\overline{B}$**) (24)**

2- The case where n = 4 is treated by (Ide. N 1996).

4-2- Remark :

It is so easy to use the brackets of complex numbers to treat other operations such as product of a matrix by a real number, λ, or the addition of two matrices. This can be done by successive multiplication of all complex number in the brackets by λ, and adding the enfaces complex numbers in th brackets of the matrices respectively.

5- Application :

There are many practical examples which show that, by using this method of brackets of complex numbers instead of the classic matricial method (in particular, the multiplication and the inverse of matrices), we need less operations. For example, if we take the quaternion of Hamilton given below for a (4,4) matricial form (De Casteljau. P, 1987) :

$$q = \begin{bmatrix} d & -a & -b & -c \\ a & d & -c & b \\ b & c & d & -a \\ c & -b & a & d \end{bmatrix} \qquad \textbf{(25)}$$

then, we write q in the brackets form as :

q = [{A, B}, { 0, -B }, {0 , B }, {A, 0 }] **(26)**

where:

A = d +i a ; B = b + i c **(27)**

By using this method for multiply two quaternions , it is sufficient to use the product of the form: {A, B}.{C, D }, i.e; we need four products of complex numbers only or 16 product operations of real numbers instead of 64 operations when we use the usual product matricial form.

<u>5-1-Numerical Examples (by using computer)</u> <u>and conclusion</u>:

1- We take the element of the matrix (3,3) :

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$$

and by using a program in Fortran we find the invers of this matrix by using this method :

$$\begin{pmatrix} -11.000010 & 2.000001 & 2.000001 \\ -4.000001 & 4.768372E-07 & 1.000000 \\ 6.000003 & -1.000000 & -1.000001 \end{pmatrix}$$

and we find by using the classical method of inverse of matrix :

$$\begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$$

2- For the matrix (3,3) :

$$\begin{pmatrix} 1.2 & 2.5 & 3 \\ 1.25 & 6 & 4.5 \\ 3.24 & 1.2 & 2.1 \end{pmatrix}$$

and by using a program in Fortran we find the invers of this matrix by using this method :

$$\begin{pmatrix} -4.708188E-1 & 1.078959E-1 & 4.413927E-1 \\ -7.817556E-1 & 4.708189E-1 & 1.078959E-1 \\ 1.173124 & -4.355075E-1 & 2.664703E-1 \end{pmatrix}$$

and we find by using the classical method of inverse of matrix :

$$\begin{pmatrix} -0.470819028 & 0.107895956 & 0.441392839 \\ -0.781755251 & 0.470819028 & 0.107895956 \\ 1.17312331 & 0.435507317 & 0.266470317 \end{pmatrix}$$

we find that, the difference between this method and the classical method is very small.

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