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Maximal Elements and Prime Elements in Lattice Modules

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ABSTRACT

Let L be a multiplicative lattice, and let M be an L-module. An element $N \neq I_M$ of M is called a maximal element if $\left|[N, I_M]\right| = 2$. It is called a prime element, if $\forall \lambda \in L, X \in M$ such that $\lambda X \leq N$ then either $X \leq N$ or $\lambda \leq (N : I_M)$.

In this paper we study the relationship between the maximal (prime) elements of M and the maximal (prime) elements of L. We show that, if L is a local lattice and the greatest element of M is weak principal, then M is local .

Then we define the Jacobson radical of M and denote it by J(M) and we study its relationship with the Jacobson radical of L (J(L)) .

Afterwards, we define the semiprime element in a lattice module M, and we show that the definitions of prime element and semiprime element are equivalent when the greatest element of M is multiplication and we study the properties equivalent to the properties of prime element in lattice module .

Also, we defined the integral module and we proved that if $\{N_i\}_{i\in I}$ is a chain (ascending or descending) of prime elements of M, then $\underset{i\in I}{\wedge} N_i$ is a prime element of M. Moreover, if I_M is compact, then $\underset{i\in I}{\vee} N_i$ is a prime element of M.

Key words: Multiplicative lattice, lattice module, maximal element, prime element, Jacobson radical .

العناصر الأعظمية والعناصر الأولية في المقاسات الشبكية

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الملخص

لتكن L شبكة ضربية و M مقاساً فوقها وليكن N عنصراً ما من M بحيث $N \neq I_M$. نقول عـن N إنه أعظمي في M إذا كان $\left|[N, I_M]\right| = 2$ ، ونقول عن N: إنه أولي في M إذا كان من أجـل أي عنـصر λ مـن L وأي عنـصر X مـن M بحيـث $\lambda X \leq N$ فإنـه إمـا $X \leq N$ أو $\lambda \leq (N : I_M)$.

درسنا في هذا البحث العلاقة بين العناصر الأعظمية (الأولية) في المقاس الـشبكي M و العناصـر الأعظمية (الأولية) في الشبكة الضربية L، وبرهنا أنه إذا كانت الـشبكة الـضربية L موضـعية وكـان العنصر الأكبر في M رئيسياً ضعيفاً فإن M يكون موضعياً أيضاً. كما عرفنا جذر جاكبسون في المقـاس M(J(M)) ودرسنا علاقته بجذر جاكبسون في الشبكة L(J(L)).

إضافة إلى ذلك فقد عرفنا العنصر نصف الأولي في المقاس الشبكي وبينا متى يكون العنصر نـصف الأولي في M عنصراً أولياً. ثم درسنا الخواص المكافئة لمفهوم العنصر الأولي فـي المقـاس الـشبكي وعرفنا المقاس التام وبرهنا أنه إذا كانت $\{N_i\}_{i \in I}$ متتالية (متزايدة أو متناقصة) من العناصر الأوليـة في M فإن $\underset{i \in I}{\wedge} N_i$ هو عنصر أولي في M، وإذا كان I_M متراصاً في M فإن $\underset{i \in I}{\vee} N_i$ يكون عنصراً أولياً أيضاً في M.

الكلمات المفتاحية: الشبكة الضربية، مقاس شبكي، عنصر أعظمي، عنـصر أولـي، جذر جاكبسون.

1. Introduction

A major step was taken by R. P. Dilworth [1] in giving an abstract formulation of the ideal theory of commutative rings . He introduced the concept of the multiplicative lattice, defined the notion of a principal element as a generalization to the notion of a principal ideal and defined the Noether lattice . Then E. W. Johnson and J. A. Johnson [2,3] introduced and studied Noetherian lattice modules and hence most of Dilworth's ideas and methods were extended .

In this paper we study the relationship between the maximal (prime) elements of an L-module M and the maximal (prime) element of the lattice L. So, most of the results of maximal (prime) submodules obtained in [7] are generelized to the lattice modules .

In this section, we present some definitions and terminolygy which are used in this paper. By a multiplicative lattice we shall mean a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the greatest element I of the lattice is an identity for multiplication .

Let L be a multiplicative lattice and let M be a complete lattice . Recall that M is an L-module ([2], definition 2.2) in case there is a multiplication between elements of L and M, denoted by aA for a in L and A in M, which satisfies: (i) (ab)A=a(bA); (ii) $(\vee_\alpha a_\alpha)(\vee_\beta B_\beta) = \vee_{\alpha,\beta} a_\alpha B_\beta$; (iii) IA = A; and (iv) oA = 0 ; for all a , a_α , b in L and for all A, B_β in M. We shall denote the greatest element of M by I_M .

Let M be an L-module . For a in L and for A, B in M,

(i) (A : a) denotes the largest C of M such that $aC \le A$; and,

(ii) (A : B) denotes the largest c of L such that $cB \le A$.

An element A in M is called weak meet principal if (B : A) A = $B \wedge A$ for all B in M; A is called weak join principal if bA : A = $b \vee (0 : A)$ for all b in L; and, A is weak principal if A is both weak meet principal and weak join principal .

An element B in M is called a multiplication element if for every element $D \le B$, there exists an element b of L such that D = bB .

Remark 1.1. Let M be an L-module and let A be an element of M. Then A is a weak meet principal of M if and only if A is multiplication .

Let W be a lattice. Let A, B be elements of W such that $A \le B$. Then the set $\{X \in W \setminus A \le X \le B\}$ is a sublattice of W which will be denoted by [A , B] ([2], Definition 2.7) .

Remark 1.2. ([2], remark 2.8) Let M be an L-module. Let A and B be elements of M such that $A \le B$. Then [A , B] is "naturally" an L-module .

For results and definitions which are not given in the paper, the reader is referred to [1,2,3] and [4] .

2. The Maximal Elements in Lattice Modules

Definition 2.1. An element $N \neq I_M$ of M is called a maximal element if for every element B of M such that $N \le B$, then either N = B or B = I_M . We shall express to the maximal element N of M by $|[N , I_M]| = 2$.

Remark 2.1. In general case , the lattice module M may not have a maximal element . The following example shows that .

Example 2.1. Let N be the set of the natural numbers and let $L = N \vee \{I\}$ with $I > n$ for each n of N. L is a complete lattice with natural ordering of N. We make L into a multiplicative lattice by defining $x.y = x \wedge y$, $\forall\, x, y \in L$. L does not have a maximal element by regarding L as an L-module .

Lemma 2.1. Let M be an L-module and let B be an element of M. If the greatest element I_M of M is weak principal, then the L-module [B, I_M] of M is isomorphic to the submodule $[(B : I_M) , I_L]$ of L .

Proof. Let the map $\varphi : [(B : I_M) , I_L] \to [B , I_M]$ be defined by $\varphi(\lambda) = \lambda I_M$. If $\lambda \in [(B : I_M) , I_L]$, then $(B : I_M) \le \lambda$ and hence $(B : I_M)\, I_M \le \lambda I_M$. Since I_M is weak principal it follows that $B = B \wedge I_M \le \lambda I_M$. Thus $\varphi(\lambda) = \lambda I_M \in [B , I_M]$. Let λ_1 , λ_2 be elements of $[(B: I_M), I_L]$ and assume that $\varphi(\lambda_1) = \varphi(\lambda_2)$. Then $\lambda_1 I_M = \lambda_2 I_M$. Hence $(\lambda_1 I_M : I_M) = (\lambda_2 I_M : I_M)$. Since I_M is weak principal, it follows that $\lambda_1 \vee (0 : I_M) = \lambda_2 \vee (0 : I_M)$ and since $\lambda_1, \lambda_2 \geq (B : I_M)$; therefore $\lambda_1 = \lambda_2$. Thus φ is one–to–one. To see that φ is onto, let X be an element of $[B , I_M]$, then $X \geq B$ and hence $(X : I_M) \geq (B: I_M)$. Thus $(X: I_M)$ is an element of $[(B : I_M) , I_M]$. Applying φ we obtain $\varphi(X: I_M) = (X: I_M)I_M = X \wedge I_M = X$, since I_M is weak principal and consequently φ is onto. Since φ is clearly order preserving, we have that φ is a lattice isomorphism of $[(B: I_M) , I_L]$ onto $[B, I_M]$.

Remark2.2. The inverse of the map φ is $\varphi^{-1}:[B,I_M] \to [(B:I_M), I_L]$ where it is defined by $\varphi^{-1}(X) = (X : I_M)$.

Proposition 2.1. Let M be an L-module in which the greatest element I_M is weak principal and let a be an element of L. Then aI_M is a maximal element of M if $a \vee (0: I_M)$ is a maximal element of L.

Proof. We have that $[a \vee (0: I_M) , I_L] = [(aI_M: I_M), I_L] \cong [aI_M, I_M]$ by Lemma 2.1. Thus $2 = |\, [a \vee (0: I_M), I_M] \,| \Leftrightarrow |\, [aI_M, I_M] \,| = 2$.

Lemma 2.2. Let M be an L-module in which the greatest element I_M is multiplication and let N be an element of M. If $(N: I_M)$ is a maximal element of L then N is a maximal element of M.

Proof. Since $(N: I_M)$ is maximal in L. Then $(N: I_M) \neq I$, thus $N \neq I_M$. Let B be an element of M such that $N \leq B$; therefore $(N: I_M) \leq (B: I_M)$; but $(N: I_M)$ is maximal in L. Then either $(N: I_M) = (B: I_M)$ or $(B: I_M) = I_L$. Hence, either $(N: I_M)I_M = (B: I_M)I_M$ or $(B : I_M) I_M = I_M$; Since I_M is multiplication, consequently I_M is weak meet principal by remark 1.1 and hence, either $N \wedge I_M = B \wedge I_M$ or $B \wedge I_M = I_M$; i.e., either $N = B$ or $B = I_M$. Thus N is a maximal element of M.

Proposition 2.2. Let M be an L-module in which the greatest element I_M is weak principal and let N be an element of M. Then N is a maximal element of M if and only if $(N: I_M)$ is a maximal element of L.

Proof. Since I_M is weak principal, we obtain $[(N: I_M), I_L] \cong [N, I_M]$ by lemma 2.1. Thus N is a maximal element of M if and only if $(N: I_M)$ is a maximal element of L.

Proposition 2.3. Let M be an L-module in which the greatest element I_M is multipication and let $N \neq I_M$ be an element of M. Then N is a maximal element of M if and only if $aI_M \vee N = I_M$, $\forall a \in R$ such that $aI_M \npreceq N$.

Proof. Let $N \neq I_M$ be a maximal element of M. Then for any element B of M such that $B \npreceq N$, we have $I_M = B \vee N$. Sine I_M is multiplication, it follows that evry element B of M has the form $B = a I_M$; $a \in L$. Thus $I_M = a I_M \vee N$.

Conversely, suppose that $aI_M \vee N = I_M$, $\forall a \in R$ such that $aI_M \npreceq N$ and let B be an element of M such that $B > N$. Then, there exists an element A of M such that $A \leq B$, $A \npreceq N$. Since I_M is a multiplication element of M, we get $A = x I_M$ for some element x of L. Hence $x I_M \npreceq N$. But $x I_M \vee N = I_M$, $\forall x \in L$ such that $x I_M \npreceq N$, so, $I_M = x I_M \vee N \leq B$. Thus $B = I_M$, which proves that N is a maximal element of M .

Proposition 2.4. ([6], proposition (2-3)) Let M be an L-module in which the greatest element I_M is compact . Then for every element $X \neq I_M$ of M, there exists a maximal element N of M such that $X \leq N$.

Definition 2.2. An L-module M is called a local module if it has only one maximal element .

Lemma 2.3. Let L be a local lattice and let M be an L-module in which the greatest element I_M is weak principal . Then M is a local module .

Proof. Since I_M is weak principal, we obtain that $[(0: I_M), I_L] \cong [0, I_M] = M$, by Lemma 2.1. Since L is local, we have $[(0: I_M), I_L]$ is local. Then M is a local module by the isomorphism .

Definition 2.3. Let J(M) denote the intersection of the maximal elements of the module M. Then J(M) is called the Jacobson radical of M .

$J(L)$ is called the Jacobson radical of L and it is the intersection of the maximal elements of L .

Now, we study the relationship between the Jacobson radical of M and the Jacobson radical of L, where the greatest element of M is weak principal .

A multiplicative lattice is said to be semi-local if it has only finite maximal elements.

Corollary 2.1. Let L be a semi-local lattice and let M be an L-module in which the greatest element I_M is weak principal. Then $J(M) = J(L)I_M$.

proof. It follows from proposition 2.1, and ([5] , proposition 4.2).

Proposition 2.5. Let M be an L-module in which the greatest element I_M is weak principal. Suppose that the annihilator of $I_M(0: I_M)$ is contained in the Jacobson radical of L. Then $J(M) = J(L)I_M$.

Proof. Let a be an element of L such that $a \leq J(L)$ and we shall prove that $aI_M \leq J(M)$. Now, suppose that $aI_M \nleq J(M)$. Then there exists a maximal element N of M such that $aI_M \nleq N$. since I_M is weak principal, I_M is multiplication and hence $N = xI_M$ for some x of L. Therefore $x \vee (0: I_M)$ is a maximal element of L (proposition 2.1). Since $aI_M \nleq xI_M = N$, we get $a \nleq x \vee (0: I_M)$, which is a contrary that $a \leq J(M)$ then $aI_M \leq J(M)$, i.e., $a \leq (J(M): I_M)$. Therefore $J(L) \leq J(M): I_M$. Thus $J(L)\, I_M \leq (J(M): I_M)\, I_M = J(M) \wedge I_M = J(M)$.

Conversely, let A be an element of M such that $A \leq J(M)$. Since I_M is weak meet principal. Then I_M is multiplication i.e., $A = aI_M$ for some a of L. Now, we shall prove that $A \leq J(L)I_M$. Suppose that $aI_M \nleq J(L)I_M$. Therefore, $a \nleq (J(L)I_M):I_M$ and hence $a \nleq J(L) \vee (0: I_M) = J(L)$. Then there exists a maximal element p of L such that $a \nleq p$. Therefore $aI_M \nleq pI_M$. Since I_M is weak principal and $(0: I_M) \leq J(L)$, we get pI_M is a maximal element of M (proposition 2.1) and hence $aI_M \nleq J(M)$, a contradiction. Then $A = aI_M \leq J(L)\, I_M$; hence $J(M) \leq J(L)I_M$. Therefore $J(M) = J(L)I_M$.

Proposition 2.6. Let M be an L-module in which the greatest element I_M is compact and let B be an element of M. Then the following statements are equivalent :

(i) $B \leq J(M)$

(ii) $\forall D \in M$ & $D \vee B = I_M \Rightarrow D = I_M$

Proof. (i)$\Rightarrow$(ii): Suppose that $D \neq I_M$. Then there exists a maximal element N of M such that $D \leq N$ (proposition 2.4). Since $B \leq J(M)$, we get $D \vee B \leq N \neq I_M$, a contradiction. Thus $D = I_M$.

(ii) $\Rightarrow$(i): Let N be a maximal element of M . If $B \not\leq N$. Then $N \vee B > N$. Therefore $N \vee B = I_M$ and hence $N = I_M$, a contradiction. Thus $B \leq N$.

3. The Prime Element of Lattice Modules

Definition 3.1. An element $N \neq I_M$ of M is called a prime element if whenever $\lambda X \leq N$, for $\lambda \in L$ and $X \in M$ implies that either $X \leq N$ or $\lambda \leq (N : I_M)$.

Definition 3.2. An element $B \neq I_M$ of M is called a semiprime element if, $\forall$ a,b$\in$L such that $abI_M \leq B$, then either $aI_M \leq B$ or $bI_M \leq B$.

Lemma 3.1. Let M be an L-module. Then every prime element of M is semiprime .

Proof. Let N be an element of M and suppose that $abI_M \leq N$ and $aI_M \not\leq N$, $\forall$ a, b$\in$L. Then, there exists $A \leq I_M$ such that $abA \leq N$ and $aA \not\leq N$. Since N is a prime element in M, it follows that $bI_M \leq N$; i.e., N is a semiprime element .

Proposition 3.1. Let $N \neq I_M$ be an element of an L-module M. Then the following statement are equivalent :

(i) N is a semiprime element .

(ii) For any two elements A and B of M, if $(A: I_M)(B: I_M) \leq (N: I_M)$, then either $(A: I_M) \leq (N : I_M)$ or $(B: I_M) \leq (N: I_M)$.

Proof. (i)$\Rightarrow$(ii): Suppose that $(A: I_M)(B: I_M) \leq (N: I_M)$ and $(B:I_M) \not\leq (N:I_M)$. Then there exists an element b of L such that $b \leq (B: I_M)$ and $b \not\leq (N: I_M)$ and hence $bI_M \leq B$. We shall prove that $(A: I_M) \leq (N: I_M)$. Let $a \leq (A: I_M)$, then $ab \leq (A: I_M)(B: I_M) \leq (N: I_M)$. Therefore $abI_M \leq N$. Since N is a semiprime element and $b \not\leq (N: I_M)$, it follows that $a \leq (N: I_M)$. That is, $(A: I_M) \leq (N: I_M)$.

(ii)$\Rightarrow$(i): Suppose that $abI_M \leq N$; then $(abI_M : I_M) \leq (N: I_M)$; but $(aI_M: I_M)(bI_M: I_M) \leq (abI_M: I_M)$; therefore $(aI_M: I_M)(bI_M: I_M) \leq (N: I_M)$. So, by (ii), either $(aI_M: I_M) \leq (N: I_M)$ or $(bI_M: I_M) \leq (N: I_M)$. But $a \leq (aI_M: I_M)$ and $b \leq (bI_M: I_M)$; thus either $aI_M \leq N$ or $bI_M \leq N$.

Proposition 3.2. Let M be an L-module in which the greatest element I_M is multiplication and let $N \neq I_M$ be an element of M . Then N is prime if and only if it is semiprime .

Proof. Suppose that N is a semiprime element and let $aX \leq N$, where X of M and a of L; therefore $(aX: I_M) \leq (N: I_M)$. However, $(aI_M:I_M)(X: I_M) \leq (aX: I_M)$, which implies that $(aI_M: I_M)(X: I_M) \leq (N: I_M)$. Using proposition 3.1, we get that either $(aI_M: I_M) \leq (N: I_M)$ or $(X:I_M) \leq (N:I_M)$. Hence, either $(aI_M: I_M)I_M \leq (N: I_M)I_M$ or $(X: I_M)I_M \leq (N: I_M)I_M$. Since I_M is multiplication element; I_M is weak meet principal by remark 1.1 and hence, either $aI_M \leq N$ or $X \leq N$; i.e., N is a prime element of M .

The converse follows from lemma 3.1.

Definition 3.3. An L-module M is called integral module, if and only if $I_M \neq 0$ and the zero element 0 of M is a prime element.

Lemma 3.2. An element N of an L-module M is prime if and only if $[N , I_M]$ is an integral module .

Proof. Obvious .

Proposition 3.3. Let $N \neq I_M$ be an element of an L-module M. Then the following statements are equivalent .

(i) N is a prime element of M

(ii) For every element B of M such that B>N, we have $(N:B)=(N:I_M)$.

(iii) $(N:X)=(N:I_M)$ for every X of M such that $X \nleq N$.

(iv) $N=(N:a)$ for every element a of L such that $(N:I_M) < a$.

Proof. (i)$\Rightarrow$(ii): It is obvious that $(N:I_M) \leq (N:B)$ and we must show that $(N:B) \leq (N:I_M)$. Let $a \leq (N:B)$, then $aB \leq N$. Since N is prime and $B \nleq N$, it follows that $a \leq N:I_M$, which proves that $(N:B) \leq (N:I_M)$. Thus $(N: B)=(N: I_M)$.

(ii)$\Rightarrow$(iii): Let X be an element of M such that $X \not\leq N$ and let Y = $N \vee X$. So, we have (N:Y)= N: ($N \vee X$)= (N:X). But (N:Y) = (N: I_M) by (ii). Hence (N: I_M) = (N: X) for every $X \not\leq N$.

(iii)$\Rightarrow$(iv): Let a be an element of L such that a>(N:I_M) and we shall prove that N = (N:a) . It is abvious that $N \leq$(N:a). On the other hand, let X be an element of M such that $X \leq$(N:a), then $aX \leq N$ and hence $a \leq$(N:X). Suppose that $X \not\leq N$. Then, by (iii) we have (N:X) = (N:I_M) and hence $a \leq$(N:I_M), a contradiction. Thus $X \leq N$ and N= (N:a).

(iv)$\Rightarrow$(i): Let $rX \leq N$ and $r \not\leq$(N: I_M). Define a=(N: I_M)$\vee$r. Then ax = (N: I_M)$X \vee rX \leq N$ and hence $X \leq$(N:a) which equal to N by (iv). Consequently $X \leq N$ and N is a prime element .

Corollary 3.1. Let M be an L-module. Then every maximal element of M is prime .

Proof. Since I_M is the only element of M properly containing every maximal element of M . Employ (proposition 3.3 (ii)) .

Corollary 3.2. Let M be an integral L-module. Then (0:X) = (0:I_M) for every element $X \neq 0$ of M .

Proof. Since M is an integral L-module, i.e., the zero element 0 of M is prime. By prposition 3.3(ii) for N = 0, it follows that (0:X) = (0:I_M) .

Corollary 3.3. Let N be a prime element of an L-module M and let a be an element of L. Then either (N:a) = N or (N:a) = I_M.

Proof. Since a(N:a)$\leq$N. By the definition of the prime element, we have $a \leq$(N:I_M) or (N:a)$\leq$N. Consequently either (N:a) = I_M or (N:a) = N .

Proposition 3.4. Let $\{N_i\}_{i \in I}$ be a chain (ascending or descending) of prime elements of an L-module M . Then:

(i) $\bigwedge_{i\in I} N_i$ is a prime element of M .

(ii) If the greatest element I_M of M is compact. Then $\bigvee_{i\in I} N_i$ is a prime element of M .

Proof. (i) Let $N_1 \leq N_2 \leq N_3 \leq \ldots\ldots \leq N_i \leq \ldots$.. be an ascending chain of elements of M. It is clear that $\bigwedge_{i\in I} N_i \neq I_M$. Let $rX \leq \bigwedge_{i\in I} N_i$, for $r \in L$ and

$X \in M$ and let $X \not\leq \bigwedge_{i\in I} N_i$. Thus $X \not\leq N_j$ for some $j \in I$, but $rX \leq N_j$, consequently $r \leq (N_j : I_M)$. Next, let $N_i \neq N_j$. Then either $N_i < N_j$, accordingly $X \not\leq N_i$ and $rX \leq N_i$, so we have $r \leq (N_i : I_M)$, or $N_j < N_i$ and hence $r \leq (N_j : I_M) \leq (N_i : I_M)$. Thus $r \leq \bigwedge_{i\in I} (N_i : I_M) = (\bigwedge_{i\in I} N_i) : I_M$ which proves that $\bigwedge_{i\in I} N_i$ is a prime element of M .

To prove **(ii)**, note that $\bigvee_{i\in I} N_i \neq I_M$, since I_M is compact . Now, let $rX \leq \bigvee_{i\in I} N_i$ and $X \not\leq \bigvee_{i\in I} N_i$. Then $rX \leq N_j$ for some $j \in I$, but $X \not\leq N_j$. Accordingly $r \leq (N_j : I_M) \leq ((\bigvee_{i\in I} N_i) : I_M)$, which proves that $\bigvee_{i\in I} N_i$ is a prime element of M .

The proof for the descending chain is analogous .

Next we study the relationship between the prime elements of an L-module M and the prime elements of L .

Proposition 3.5. Let M be an L-module in which the greatest element I_M is weak principal and let a be an element of L. Then aI_M is a prime element of M if and only if $a \vee (0:I_M)$ is a prime element of L .

Proof. By proposition 2.1, we have that $[a \vee (0: I_M), I_L] = [(aI_M : I_M), I_L] \cong [aI_M, I_M]$, which proves that aI_M is a prime element of M if and only if $a \vee (0:I_M)$ is a prime element of L .

Proposition 3.6. Let N be an element of an L-module M . If N is a prime element of M. Then $(N:I_M)$ is a prime element of L.

Proof. Since N is a prime element of M; i.e., $N \neq I_M$ and hence $(N:I_M) \neq I$. Let a and b be elements of L such that $ab \leq (N:I_M)$ and suppose that $b \not\leq (N: I_M)$. This means that $abI_M \leq N$ and $bI_M \not\leq N$; but N is a prime element of M; therefore $a \leq (N:I_M)$ which implies that $(N:I_M)$ is prime .

Remark 3.1. The following example shows that the converse of the above proposition is not true in general .

Example 3.1. Let Z be the ring of integers, and let Z×Z be the Z-module. Suppose that L = L(Z) is the set of all ideals of Z and

$M = L(Z\times Z)$ is the set of all submodules of $Z\times Z$. Let $N = (4,0)Z$ be an element of M. Clearly $(N: I_M) = ((4,0)Z: Z\times Z) = 0$, because if $\lambda(1,1) = (\lambda,\lambda) \in (4,0)Z$ for some λ of Z. Then $\lambda = 0$, which means that $(N:I_M)$ is a prime element of L. In the other hand, we have $2Z(2,0)Z \leq N$, but $(2,0)Z \not\leq N$ and $2Z \not\leq (N:I_M)$. Thus N is not a prime element of M .

Proposition 3.7. Let M be an L-module in which the greatest element I_M is weak principal and let N be an element of M. Then N is a prime element of M if and only if $(N:I_M)$ is a prime element of L .

Proof. Since I_M is weak principal, we obtain $[(N: I_M), I_L] \cong [N, I_M]$, by Lemma 2.1. Thus N is a prime element of M if and only if $(N:I_M)$ is a prime element of L .

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