On $(\Delta-, \nabla-, \ \mathrm{I}-)$ semipotent and the total of rings and modules

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ABSTRACT

Let M and N be two modules over a ring R. The object of this paper is the study of substructures of $hom_R(M,N)$ ***such as, radical, the singular, and co-singular ideal and the total. The new obtained results include necessary and sufficient conditions the total of a ring R to equal some ideal of R. The main result states that:***

(a) $\mathrm{Tot}\,[M,N] = \Delta\,[M,N]$ ***if and only if,*** $[M,N]$ ***is*** $\Delta-$***semipotent, for any*** $M, N \in mod-R$.

(b) $\mathrm{Tot}\,[M,N] = \nabla\,[M,N]$ ***if and only if,*** $[M,N]$ ***is*** $\nabla-$***semipotent, for any*** $M, N \in mod-R$.

(c)***If for a module*** $M \in mod-R$, $\Gamma(M) = \{0\}$ ***then*** $\mathrm{Tot}\,[M,N] = I\,[M,N]$ ***if and only if,*** $[M,N]$ ***is*** $I-$***semipotent, for any*** $N \in mod-R$.

(d)***If for a module*** $N \in mod-R$, $\Gamma(N) = \{0\}$ ***then*** $\mathrm{Tot}\,[M,N] = I\,[M,N]$ ***if and only if,*** $[M,N]$ ***is I- semipotent, for any*** $M \in mod-R$.

Also, we characterize the modules V and W for which :

(a) $\mathrm{Tot}\,[M,W] = \Delta\,[M,W]$ ***and*** $\mathrm{Tot}\,[V,N] = \Delta\,[V,N]$ ***for all*** $M, N \in mod-R$.

(b) $\mathrm{Tot}\,[M,W] = \nabla\,[M,W]$ ***and*** $\mathrm{Tot}\,[V,N] = \nabla\,[V,N]$ ***for all*** $M, N \in mod-R$.

(c) $\mathrm{Tot}\,[M,W] = I\,[M,W]$ ***and*** $\mathrm{Tot}\,[V,N] = I\,[V,N]$ ***for all*** $M, N \in mod-R$.

Key Words: $(\Delta-, \nabla-, \ \mathrm{I}-)$ *Semipotent Rings,* I_0-*Rings,* I_0-*modules, The total, Jacobson radical, (co) singular ideal, Endomorphism rings,* $hom_R(M,N)$.

حول الحلقات والمودولات $(\Delta-, \nabla-, -I)$ شبه الجامدة والتوتال

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الملخص

ليكن M, N مودولين فوق الحلقة R. إن الغاية من هذه الورقة هو متابعة دراسة البنـى الجزئيـة للمودول $hom_R(M,N)$ مثل الأساس والمثالي المنفرد والمثالي المنفرد الثنوي والتوتـال. نتـائج جديدة تم الحصول عليها فعلى سبيل المثال تم إيجاد الشرط اللازم والكافي كي يكون التوتال لحلقـة مـا يساوي مثالياً معيناً لهذه الحلقة. النتائج الرئيسية في هذا العمل هي:

1. $\text{Tot}[M,N] = \Delta[M,N]$ **عندما وفقط عندما** $[M,N]$ **هو** $\Delta-$ **شبه جامد وذلك لأجـل أي** $M,N \in mod-R$**.**
2. $\text{Tot}[M,N] = \nabla[M,N]$ **عندما وفقط عندما** $[M,N]$ **هو** $\nabla-$ **شبه جامد وذلك لأجل أي** $M,N \in mod-R$**.**
3. **إذا كان لأجل المـودول** $M \in mod-R$ **،** $\Gamma(M) = \{0\}$ **فـإن** $\text{Tot}[M,N] = I[M,N]$ **عندما وفقط عندما** $[M,N]$ **هو** $I-$ **شبه جامد وذلك لأجل أي** $N \in mod-R$**.**
4. **إذا كان لأجل المودول** $N \in mod-R$ **،** $\Gamma(N) = \{0\}$ **فإن** $\text{Tot}[M,N] = I[M,N]$ **عندما وفقط عندما** $[M,N]$ **هو** $I-$ **شبه جامد وذلك لأجل أي** $M \in mod-R$**.**

فضلاً عن ذلك فقد وصفت المودولات V,W **التي من أجلها:**

1. $\text{Tot}[M,W] = \Delta[M,W]$ **و** $\text{Tot}[V,N] = \Delta[V,N]$ **وذلك لأجل أي** $M,N \in mod-R$**.**
2. $\text{Tot}[M,W] = \nabla[M,W]$ **و** $\text{Tot}[V,N] = \nabla[V,N]$ **وذلك لأجل أي** $M,N \in mod-R$**.**
3. $\text{Tot}[M,W] = I[M,W]$ **و** $\text{Tot}[V,N] = I[V,N]$ **وذلك لأجل أي** $M,N \in mod-R$**.**

الكلمـات المفتاحيـة: الحلقـات $(\Delta-, \nabla-, -I)$ شـبه الجامـدة، I_0-حلقـة، I_0-مودول، التوتال، أساس جاكبسون، المثالي المنفرد، حلقة التشاكلات لمودول، $hom_R(M,N)$.

1- Introduction

In this paper rings R are associative with identity unless otherwise indicated. All modules over a ring R unitary right modules. A submodule N of a *module M is said to be small in M if* $N+K \neq M$ *for any proper submodule K of M, [4]. A submodule N of a module M is said to be large (essential) in M if* $N \cap K \neq 0$ *for any nonzero submodule K of M, [4]. If* M *is an* $R-$*module, the radical of M. denoted by* $J(M)$*, is defined to be the intersection of all maximal submodules of M. Also,* $J(M)$ *coincides with the sum of all small submodules of M. It my happen that* M *has no maximal submodules in which case* $J(M)=M$*,[11]. Thus, for a ring R,* $J(R)$ *is the Jacobson radical of R. For a submodule N of a module M, we use* $N \subseteq^{\oplus} M$ *to mean that N is a direct summand of M, and we write* $N \leq_e M$ *and* $N << M$ *to indicate that N is a large, respectively small, submodule of M. If* M_R *is a module, we use the notation* $E_M = End_R(M)$ *and we write:*

$\Delta E_M = \{\alpha : \alpha \in E_M;\ Ker(\alpha) \leq_e M\}$, $\nabla E_M = \{\alpha : \alpha \in E_M;\ Im(\alpha) << M\}$ *and* $I(E_M) = \{\alpha : \alpha \in E_M;\ Im(\alpha) \subseteq J(M)\}$. *It is will known that* ΔE_M, ∇E_M *and* $I(E_M)$ *are ideals in* E_M*, [4]. It is easy to see that* $\nabla E_M \subseteq I(E_M)$. *If* M_R *and* N_R *are modules, we use* $[M,N] = hom_R(M,N)$. *Thus* $[M,N]$ *is an* $(E_N, E_M)-$*bimodule. Our main concern is about the substructures of* $hom_R(M,N)$ *and the* $(\Delta-, \nabla-, \mathrm{I}-)$ *semipotent of* $hom_R(M,N)$ *(see [14]).*

The total is a concept that was first introduced by F. Kasch 1982, 2002[5], and Y. Zhou [14] in 2009. In the study of the total, one of the interesting questions is when the total equals the Jacobson radical, the singular ideal and the co-singular ideal. In section, (2) *it is proved that* $\mathrm{Tot}(R) = I$ *if and only if, R is an I-semipotent ring and the ideal I contains no nonzero idempotents. In section* (3)*, it is proved that a quasi-projective module P is a semipotent if and only if,* E_P *is anI-semipotent ring. Interesting corollaries are obtained in this section. In particular,* $\mathrm{Tot}\,[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in \mathrm{Tot}\,(E_M) \text{ for all } \beta \in [N,M]\}$.

In section (5)*, it is proved that* $[M,N]$ *is* $\Delta-$*semipotent if and only if,* $\text{Tot}\,[M,N] = \Delta[M,N]$. *Also, in this section we characterize the modules* V *and* W *for which* $\text{Tot}\,[V,N] = \Delta\,[V,N]$ *and* $\text{Tot}\,[M,W] = \Delta\,[M,W]$ *for all* $N, M \in mod-R$. *The main result states that* E_V *is* $\Delta-$*semipotent if and only if,* $\text{Tot}\,[V,N] = \Delta\,[V,N]$ *for all* $N \in mod-R$. *Also, in this section it is proved that* $[M,N]$ *is* $\nabla-$ *semipotent if and only if,* $\text{Tot}\,[M,N] = \nabla\,[M,N]$. *Also, in this section, we characterize the modules V and W for which* $\text{Tot}\,[V,N] = \nabla\,[V,N]$ *and* $\text{Tot}\,[M,W] = \nabla\,[M,W]$ *for all* $M, N \in mod-R$. *The main result states that* E_V *is* $\nabla-$ *semipotent if and only if,* $\text{Tot}\,[V,N] = \nabla\,[V,N]$ *for all* $N \in mod-R$ *if and only if,* $\text{Tot}\,[M,V] = \nabla\,[M,V]$ *for all* $M \in mod-R$.

2- (*I* -) Semipotent Rings

Recall that a ring R is a semipotent ring, also called I_0-*ring by Nicholson [6], Hamza [3], if every principal left (resp. right) ideal not contained in J(R) contains a nonzero idempotent. Examples of these rings include*: (a) *Exchange ring (see [8, Proposition 1.9], a ring R is exchange if for each* $a \in R$ *there exists* $e^2 = e \in R$ *such that* $a - e \in (a^2 - a)R$*).* (b) *Endomorphism rings of injective modules (see [6, Proposition 1.4]).* (c) *Endomorphism ring of regular modules in the sense Zelmanowitez [15], (see [3, Corollary 3.6]). Let N and L are submodules of a module* M_R. N *is called a supplement of* L *in* M *if* $N + L = M$ *and* $N \cap L$ *is small in N. N is said to be a supplement submodule of M if N is a supplement of some submodule of M.*

Theorem 2.1. *For any ring R the following conditions are equivalent:*

(1) *R is a semipotent ring.*

(2) *For any* $a \in R$ *there exists* $0 \neq x \in R$ *such that* R/axR *has a projective cover (as a right R-module).*

(3) *For any* $a \in R$ *there exists* $0 \neq x \in R$ *such that* axR *has a supplement in* R_R *(as a right R- module) which also has a supplement.*

Proof. $(1)\Rightarrow(2)$. *Let* $a\in R$, *if* $a\in J(R)$ *then for any* $x\in R$ *the natural epimorphism* $R\to R/axR$ *is a projective cover of* R/axR. *Suppose* $a\notin J(R)$ *then* $x=xax$ *for some* $x\neq 0$. *Let* $e=ax$, *then* $e\neq 0$ *is an idempotent in* R *and* $axR=eR$. *Since* $(1-e)R\cong R/axR$ *we have* R/axR *has a projective cover.* $(2)\Rightarrow(3)$ *follows by [2, Proposition 1.4].* $(3)\Rightarrow(1)$. *Let* $a\in R$, $a\notin J(R)$ *then there exists* $y\in R$ *such that* ayR *has a supplement L which has also a supplement, by [2, Proposition 1.4],* ayR *has a supplement* K *which is a direct summand of* R. *Thus* $R=ayR+K$ *and by [2, Proposition 1.2] there exists a direct summand* eR *of* R, $eR\subseteq ayR\subseteq aR$, *where* e *is a non-zero idempotent of R. Thus, R is a semipotent ring.*

If T is a left ideal or right ideal of R, we say that idempotents lift module T if, whenever $a^2-a\in T$, $a\in R$ *there exists* $e^2=e\in R$ *such that* $e-a\in T$. *Nicholson in [9] gave an example of a commutative semipotent ring where idempotents do not lift modulo* $J(R)$, *(See [9, Example 25]). Therefore, we extend this notion as follows:*

Lemma 2.2. *The following statements are equivalent for a right ideal T of R:*

(1) *If* $a^2-a\in T$ *there exists* $e^2=e\in aR$, $e\notin T$.

(2) *If* $a^2-a\in T$ *there exists* $e^2=e\in Ra$, $e\notin T$.

Proof. Suppose (1) *holds. Then* $e^2=e=ax$ *for some* $x\in R$ *and* $e\notin T$. *We put* $y=xax$ *then* $f=ya$ *is an idempotent of R and* $f\in Ra$ *and* $f\notin T$. $(2)\Rightarrow(1)$ *is analogous.*

We say that a right ideal T of a ring R is weakly lifting, or that idempotents lift weakly modulo T, if the conditions in lemma 2.2 are satisfied. The definition for left ideals is analogous.

Proposition 2.3. *For any ring R the following conditions are equivalent:*

(1) *R is a semipotent ring.*

(2) $\overline{R}=R/J(R)$ *is a semipotent ring and* $J(R)$ *is weakly lifting.*

Proof. Suppose (1) *holds. It is clear that* $R/J(R)$ *is a semipotent ring. Let* $a^2-a\in J(R)$ *then* $a\in R$, $a\notin J(R)$ *and* $x=xax$ *for some*

$0 \neq x \in R$. *Thus,* $e = ax \in aR$ *and* $0 \neq e^2 = e \in R$. *This shows that* $J(R)$ *is a weakly lifting.* $(2) \Rightarrow (1)$. *Let* $a \in R$, $a \notin J(R)$, *since* $\overline{R}$ *is semipotent there exists* $0 \neq \overline{f}^2 = \overline{f} \in \overline{a}\overline{R}$ *therefore* $f = ar + x$ *for some* $r \in R$, $x \in J(R)$. *By our hypothesis there exists* $0 \neq e^2 = e \in fR$. *Suppose* $e = fy$ *for some* $y \in R$, *then* $e = ary + xy$. *On the other hand, since* $xy \in J(R)$ *then* $1 - xy$ *has an inverse* $b \in R$. *Thus,* $eb = aryb + xyb$, $ebe = arybe - (1-b)e$ *and* $e = earybe$. *Suppose* $g = arybe$ *then* $0 \neq g^2 = g \in aR$, *so R is semipotent.*

The semipotent rings are generalized as following:

Lemma 2.4. *[9, Lemma 19]. The following conditions are equivalent for an ideal I of a ring* R *:*

(1) *If* $T \not\subset I$ *is a right (resp. left) ideal there exists* $e^2 = e \in T \setminus I$.

(2) *If* $a \notin I$ *there exists* $e^2 = e \in aR \setminus I$ *(resp.* $e^2 = e \in Ra \setminus I$ *).*

(3) *If* $a \notin I$ *there exists* $x \in R$ *such that* $x = xax \notin I$.

Let R be a ring and I is an ideal of R, recall R is an I- semipotent [9], if the conditions in lemma 2.4, are satisfied.

Corollary 2.5. *Let I be an ideal of a ring R. If* R *is I- semipotent then* $J(R) \subseteq I$.

Proof. Suppose $J(R) \not\subset I$ *there exists* $a \in J(R), a \notin I$, *so* $x = xax \notin I$ *for some* $x \in R$. *Since* $x \neq 0$ *then* $0 \neq (ax)^2 = ax \in J(R)$ *this is a contradiction.*

Proposition 2.6. *Let I be an ideal of a ring R. The following statements are equivalent:*

(1) *R is an I- semipotent ring.*

(2) R/I *is a semipotent ring with* $J(R/I) = \overline{0}$ *and* I *is weakly lifting.*

Proof. Suppose (1) *holds. First we prove that* $J(R/I) = \overline{0}$. *Assume* $J(R/I) \neq \overline{0}$ *then there exists* $\overline{0} \neq \overline{a} \in J(R/I)$. *So* $a \in R$, $a \notin I$ *therefore* $x = xax \notin I$ *for some* $x \in R$. *Thus,* $\overline{0} \neq (\overline{a}.\overline{x})^2 = \overline{a}.\overline{x} \in J(R/I)$ *a contradiction, so* $J(R/I) = \overline{0}$. *It is clear that* R/I *is semipotent. Finally, we prove that I is a weakly lifting. Let* $a^2 - a \in I$ *then* $a \in R \setminus I$. *Since R is I-semipotent there exists* $y \in R$, $y = yay \notin I$, *so*

$0 \neq (ay)^2 = ay \in aR$, *and* $ay \notin I$. $(2) \Rightarrow (1)$. *Let* $a \in R \setminus I$ *then* $\overline{0} \neq \overline{a} \in R/I$. *Since* R/I *is semipotent and* $J(R/I) = \overline{0}$ *then* $\overline{x} = \overline{x}.\overline{a}.\overline{x}$ *for some* $0 \neq \overline{x} \in R/I$. *Since* $(ax)^2 - ax \in I$ *and I is a weakly lifting there exists* $0 \neq e^2 = e \in axR \subseteq aR$ *and* $e \notin I$, *so* R *is I- semipotent.*

Following [14], the total of a ring R *is*

$\text{Tot }(R) = \{a : a \in R;\ aR \text{ contains no nonzero idempotents}\}$

$\text{Tot }(R) = \{a : a \in R;\ Ra \text{ contains no nonzero idempotents}\}$

Y. Zhou, proved that, for a ring R, $\text{Tot }(R) = J(R)$ *if and only if R is a semipotent, [14، Theorem 2.2]. For an I- semipotent ring we have:*

Theorem 2.7. *Let I be an ideal of a ring R. The following statements are equivalent:*

(1) $\text{Tot }(R) = I$.

(2)*R is an I-semipotent ring and I contains no nonzero idempotents.*

(3) R/I *is a semipotent and* $J(R/I) = \overline{0}$ *with I contains no nonzero idempotents and I is weakly lifting.*

Proof. $(1) \Rightarrow (2)$. *It is clear that I contains no nonzero idempotents. Let* $a \in R \setminus I$ *then* $a \notin \text{Tot }(R)$. *So aR contain a nonzero idempotent. This shows that* R *is an I- semipotent ring.* $(2) \Rightarrow (1)$. *Suppose that* $\text{Tot }(R) \neq I$, *Since* $I \subseteq \text{Tot }(R)$ *there exists* $a \in \text{Tot }(R)$ *such that* $a \notin I$. *So, for some* $x \in R$, $x = xax \notin I$ *and* $0 \neq (ax)^2 = ax \in$ *aR, a contradiction.* $(2) \Leftrightarrow (3)$. *By proposition 2.6.*

3- Semipotent Modules

Let M_R *be a module and* $K \subseteq^{\oplus} M$, *then* $K \subseteq J(M)$ *if and only if* $K \cap J(M) = J(K)$. *Putting* $\Gamma(M) = \{K : K \subseteq^{\oplus} M;\ J(K) = K\}$. *Note that for any projective module P,* $\Gamma(P) = \{0\}$. *In addition to, if* $J(M) << M$ *(or M is finitely generated) for some* $M \in mod - R$ *then* $\Gamma(M) = \{0\}$.

Let M_R *be a module, letting* $I = I(E_M) = \{\alpha : \alpha \in E_M;\ Im(\alpha) \subseteq J(M)\}$.

It is clear that $I = I(E_M)$ *is an ideal in* E_M *and* $I(E_M) = \{\alpha : \alpha \in E_M;\ \beta\alpha \in I(E_M)$ *for all* $\beta \in E_M\}$ $I(E_M) = \{\alpha : \alpha \in E_M;\ \alpha\beta \in I(E_M)$ *for all* $\beta \in E_M\}$

Recall a module M_R *is a semipotent module also, called* I_0- *module [3], weakly regular module [1] if, each submodule of* M *not contained in* $J(M)$ *contains a direct summand* N *of* M, $N \notin \Gamma(M)$.

Lemma 3.1. *Let* M_R *be a semipotent module. The following holds:*

(1) *Every submodule* N *of* M *such that* $J(N) = N \cap J(M)$ *is semipotent.*

(2) *Every direct summand of M is semipotent.*

(3) *Every supplement submodule of M is semipotent.*

Proof. (1). *Let N be a submodule of M with* $J(N) = N \cap J(M)$ *and A be a submodule of N,* $A \not\subset J(N)$ *then A is a submodule of M and* $A \not\subset J(M)$. *So there exists* $K \subseteq^{\oplus} M$, $K \notin \Gamma(M)$ *and* $K \subseteq A$ *therefore* $M = K \oplus D$ *for some submodule D of M. Since* $K \subseteq A \subseteq N$ *then* $N = K \oplus (N \cap D)$. *Thus,* $K \subseteq^{\oplus} N$ *and* $K \notin \Gamma(N)$. *So N is semipotent.* (2). *Since for any direct summand H of M* $J(H) = H \cap J(M)$ *then by (1), H is semipotent.* (3). *Let N be a supplement submodule of M, by [13, 41.1] we have* $J(N) = N \cap J(M)$, *by (1), N is semipotent.*

A module P_R *is called a quasi-projective module [12] if given an epimorphism* $\beta \in [P, M]$ *and any morphism* $\alpha \in [P, M]$ *there exists* $\lambda \in E_P$ *such that* $\beta\lambda = \alpha$. *For a quasi-projective module we have the following:*

Theorem 3.2. *For any quasi-projective module* P_R *the following statements are equivalent:*

(1) *P is a semipotent module.*

(2) *For any* $\alpha \in E_P$, $\alpha \notin I(E_P)$, $Im(\alpha)$ *contains a direct summand N of P such that* $N \notin \Gamma(P)$.

(3) E_P *is an I- semipotent ring.*

Proof. $(1)\Rightarrow(2)$. *It is clear.* $(2)\Rightarrow(3)$. *Let* $\alpha\in E_P\setminus I(E_P)$ *then* $Im(\alpha)\not\subset J(P)$ *and there exists* $N\subseteq^{\oplus} P$, $N\subseteq Im(\alpha)$ *and* $N\notin\Gamma(P)$. *Let* γ *be the projection of P on to N, then* $Im(\gamma\alpha)=N$, *so there exists* $\beta\in E_P$ *such that* $\gamma\alpha\beta=\gamma$. *We* put $\mu=\beta\gamma$, *then* $\mu\alpha\mu=\mu\notin I(E_P)$. *Because if* $\mu\in I(E_P)$ *then* $\gamma=\gamma\gamma=\gamma\alpha\beta\gamma\in I(E_P)$ *that is* $N\in\Gamma(P)$ *a contradiction. So,* E_P *is I-semipotent.* $(3)\Rightarrow(1)$. *Let* A *be a submodule of P,* $A\not\subset J(P)$ *then* $A\not\subset D$ *for some maximal submodule D of P, and* $A+D=P$. *By [10, Lemma 1.1] there are* $f,g\in E_P$ *such that* $1=f+g$ *and* $Im(f)\subseteq A$, $Im(g)\subseteq D$. *It is clear that* $f\notin I(E_P)$. *By assumption there exists* $\varphi\in E_P$ *such that* $\varphi=\varphi f\varphi\notin I(E_P)$ *.Since* $(f\varphi)^2=f\varphi$ *then* $Im(f\varphi)\subseteq^{\oplus} P$, $Im(f\varphi)\subseteq^{\oplus} A$ *and* $Im(f\varphi)\notin\Gamma(P)$. *So, P is semipotent.*

Corollary 3.3. *For any quasi-projective module P the following are equivalent:*

(1) *P is a semipotent module and* $\Gamma(P)=\{0\}$.

(2) E_P *is an I- semipotent ring and* $\Gamma(P)=\{0\}$.

(3) $\mathrm{Tot}\,(E_P)=I(E_P)$.

Proof. $(1)\Leftrightarrow(2)$. *By Theorem 3.2.* $(2)\Leftrightarrow(3)$. *By Theorem 2.7 because* $\Gamma(P)=\{0\}$ *if and only if* $I(E_P)$ *contains no nonzero idempotents.*

Corollary 3.4. *Let P be a quasi-projective module with* $J(P)<<P$. *The following statements are equivalent:*

(1) *P is a semipotent module.*

(2) *For any* $\alpha\in E_P$, $\alpha\notin J(E_P)$ *there exists* $0\neq N\subseteq^{\oplus} P$, $N\subseteq Im(\alpha)$.

(3) E_P *is a semipotent ring.*

(4) $\mathrm{Tot}\,(E_P)=J(E_P)=\nabla E_P=I(E_P)$.

Proof. $(1)\Leftrightarrow(2)\Leftrightarrow(3)$. *As in Theorem 3.2، because for a quasi-projective module with* $J(P)<<P$, $J(E_P)=\nabla E_P=I(E_P)$ *by [11, Lemma 2].* $(3)\Leftrightarrow(4)$ *By [14, Theorem 2.2].*

A module P_R is called a direct-projective module [8] if, given any direct summand N of P with projection $\pi : P \to N$ and any epimorphism $\alpha : P \to N$ there exists $\beta \in E_P$ such that $\alpha.\beta = \pi$. If P is a direct-projective module then $\nabla E_P \subseteq J(E_P)$ (see [8, Theorem 3.1]). For direct projective modules we have the following:

Proposition 3.5. *Let P_R be a direct-projective module. If P is semipotent then:*

(1) *E_P is an I- semipotent ring.*

(2) $J(E_P) \subseteq I(E_P)$.

Proof. (1). *Let $\alpha \in E_P$, $\alpha \notin J(E_P)$ then there exists $N \subseteq^{\oplus} P$, $N \notin \Gamma(P)$ and $N \subseteq Im(\alpha)$. Let γ be the projection of P on to N, then $N = Im(\gamma) = Im(\gamma\alpha)$. Since P is a direct-projective there exists $\beta \in E_P$ such that $\gamma\alpha\beta = \gamma$. Putting $\mu = \beta\gamma$ then $0 \neq \mu \in E_P$, $\mu\alpha\mu = \mu$ and $\mu \notin I(E_P)$, because if, $\mu \in I(E_P)$ then $\gamma = \gamma\alpha\beta\gamma \in I(E_P)$, thus $N = Im(\gamma) \subseteq J(P)$ this means that $N \in \Gamma(P)$, a contradiction. This shows that E_P is I- semipotent.* (2). *By Corollary 2.5.*

Corollary 3.6. *Let P_R be a direct-projective module. If P is a semipotent and $J(P) << P$ then E_P is a semipotent ring.*

Proof. We have by [8, Theorem 3.1], $\nabla E_P \subseteq J(E_P)$ and by proposition 3.5, $J(E_P) \subseteq I(E_P)$. Since $J(P) << P$ then $I(E_P) = \nabla E_P$ thus $J(E_P) = \nabla E_P = I(E_P)$, so E_P is a semipotent ring.

A module P_R is called $\pi-$ projective [10] if, for any two submodules U, V of P with $P = U + V$; $E_P = [P,U] + [P,V]$. For a $\pi-$ projective modules we have the following:

Proposition 3.7. *Let P_R be a $\pi-$projective module. The following statements are equivalent:*

(1) *P is a semipotent module.*

(2) *For any* $\alpha \in E_P$, $\alpha \notin I(E_P)$ *there exists* $N \subseteq^{\oplus} P$, $N \notin \Gamma(P)$ *and* $N \subseteq Im(\alpha)$. *Proof.* $(1) \Rightarrow (2)$. *It is clear.* $(2) \Rightarrow (1)$. *Let* A *be a submodule of* P, $A \not\subset J(P)$ *then there exists a maximal submodule* M *of* P, $A \not\subset M$ *therefore* $P = A + M$. *Since* P *is a* $\pi -$ *projective there are* $\alpha, \beta \in E_P$ *such that* $1 = \alpha + \beta$ *and* $Im(\alpha) \subseteq A$, $Im(\beta) \subseteq M$.*It is clear that* $Im(\alpha) \not\subset J(P)$, *because if* $Im(\alpha) \subseteq J(P)$. *we have* $P = Im(\alpha) + Im(\beta) \subseteq J(P) + M \subseteq M \subseteq P$ *thus* $P = M$, *a contradiction. By assumption* $Im(\alpha) \subseteq A$ *contains a direct summand* N *of* P, $N \notin \Gamma(P)$. *So* P *is a semipotent module.*

Corollary 3.8. *Let* P_R *be a* $\pi -$*projective module. If* E_P *is an* $I -$*semipotent ring the following hold:*

(1) *For any* $\alpha \in E_P$, $\alpha \notin I(E_P)$ *there exists* $N \subseteq^{\oplus} P$, $N \notin \Gamma(P)$ *and* $N \subseteq Im(\alpha)$.

(2) P *is a semipotent module.*

Proof. (1). *Let* $\alpha \in E_P \setminus I(E_P)$ *then there exists* $\gamma \in E_P \setminus I(E_P)$ *such that* $\gamma = \gamma \alpha \gamma$. *Since* $0 \neq (\alpha \gamma)^2 = \alpha \gamma \in E_P$ *then* $Im(\alpha \gamma) \subseteq^{\oplus} P$, $Im(\alpha \gamma) \notin \Gamma(P)$ *and* $Im(\alpha \gamma) \subseteq Im(\alpha)$. (2). *By* (1) *and proposition 3.7.*

Proposition 3.9. *For any projective module* P_R *the following statements are equivalent:*

(1) P *is a semipotent module and* $J(P) << P$.

(2) E_P *is a semipotent ring.*

(3) *For any* $\alpha \in E_P$, $P/Im(\alpha \beta)$ *has a projective cover for some* $0 \neq \beta \in E_P$.

(4) *For any* $\alpha \in E_P$, $Im(\alpha \beta)$ *has a supplement which also has a supplement for some* $0 \neq \beta \in E_P$.

Proof. $(1) \Leftrightarrow (2)$. *By[3, Theorem 3.5].* $(2) \Rightarrow (3)$. *Suppose that* E_P *is a semipotent then by theorem 2.1 for any* $\alpha \in E_P$ *there exists* $0 \neq \beta \in E_P$ *such that* $E_P/(\alpha \beta)E_P$ *has a projective cover and by [2, Proposition 2.9]* $P/Im(\alpha \beta)$ *has a projective cover.* $(3) \Rightarrow (2)$

follows immediately from [2، Proposition 2.9] and theorem 2.1. $(3) \Leftrightarrow (4)$. *By [2, Proposition 1.4].*

Lemma 3.10. *Let M_R be a module with E_M is a semipotent ring then:*

(1) $\nabla E_M \subseteq J(E_M)$ *and* $\Delta E_M \subseteq J(E_M)$.

(2) *If* $\Gamma(M) = \{0\}$ *then* $I(E_M) \subseteq J(E_M)$.

(3) *If* $J(M) << M$ *then* $\nabla E_M = I(E_M) \subseteq J(E_M)$.

Proof. (1). *Let* $\alpha \in \nabla E_M$ *then* $Im(\alpha) << M$. *Suppose that* $\alpha \notin J(E_M)$ *then* $\beta = \beta\alpha\beta$ *for some* $0 \neq \beta \in E_M$. *Let* $\gamma = \alpha\beta$ *then* $0 \neq \gamma^2 = \gamma \in E_M$ *and* $Im(\gamma) << M$, *hence* $Im(\gamma) \subseteq Im(\alpha)$. *Therefore* $Im(\gamma) = Im(\gamma) \cap Im(1-\gamma) = 0$, *thus* $\gamma = 0$ *this is a contradiction, hence* $\alpha \in J(E_M)$ *and* $\nabla E_M \subseteq J(E_M)$. *Let* $g \in \Delta E_M$ *then* $Ker(g)$ *is large in* M. *Suppose that* $g \notin J(E_M)$ *then* $\mu = \mu g \mu$ *for some* $0 \neq \mu \in E_M$. *Let* $\nu = \mu g$ *then* $0 \neq \nu^2 = \nu \in E_M$ *and* $Ker(g) \subseteq Ker(\nu)$, *therefore* $Ker(g) \cap Im(\nu) = 0$ *thus* $Im(\nu) = 0$, *hence* $Ker(g)$ *is large in* M *and* $\nu = 0$ *this is a contradiction, hence* $g \in J(E_M)$ *and* $\Delta E_M \subseteq J(E_M)$.

(2). *Suppose that* $\Gamma(M) = \{0\}$. *Let* $\alpha \in I(E_M)$ *then* $Im(\alpha) \subseteq J(M)$. *Suppose that* $\alpha \notin J(E_M)$ *then* $\gamma = \gamma\alpha\gamma$ *for some* $0 \neq \gamma \in E_M$. *We put* $g = \alpha\gamma$ *then* $0 \neq g^2 = g \in E_M$, $Im(g) \subseteq^{\oplus} M$ *and* $Im(g) \subseteq J(M)$ *therefore* $Im(g) \in \Gamma(M) = \{0\}$, *so* $Im(g) = 0$ *and* $g = 0$ *this is a contradiction. Thus* $\alpha \in J(E_M)$. *(3). It is clear.*

It is known that for any module M_R, $\nabla E_M \subseteq I(E_M)$. *No, if* E_M *is an I- semipotent ring we have the following:*

Lemma 3.11. *Let M_R be a module with E_M is an I-semipotent ring then:* $J(E_M) \subseteq I(E_M)$ *and* $\Delta E_M \subseteq I(E_M)$.

Proof. By Corollary 2.5 we have $J(E_M) \subseteq I(E_M)$. *Let* $\alpha \in \Delta E_M$ *then* $Ker(\alpha) \leq_e M$. *Suppose* $\alpha \notin I(E_M)$ *then* $\gamma = \gamma\alpha\gamma$ *for some* $0 \neq \gamma \in E_M$, $\gamma \notin I(E_M)$. *Let* $\nu = \gamma\alpha$ *then* $0 \neq \nu^2 = \nu \in E_M$ *and*

$Ker(\alpha) \subseteq Ker(\nu)$, *therefore* $Ker(g) \cap Im(\nu) = 0$ *thus* $Im(\nu) = 0$, *so* $\nu = 0$ *this is a contradiction, hence* $\alpha \in I(E_M)$ *and* $\Delta E_M \subseteq I(E_M)$.

4. Semipotent $[M,N]$.

Following [14], let M_R, N_R *are modules and* $[M,N] = hom_R(M,N)$, *then* $[M,N]$ *is an* $(E_N, E_M)-$*bimodule.*

- *The Jacobson radical.*

$J[M,N] = \{\alpha : \alpha \in [M,N];\ \beta\alpha \in J(E_M)$ *for all* $\beta \in [N,M]\}$

$J[M,N] = \{\alpha : \alpha \in [M,N];\ \alpha\beta \in J(E_N)$ *for all* $\beta \in [N,M]\}$

Thus $J[M,M] = J(E_M)$. *In particular* $J[R,R] = J(R)$.

- *The singular ideal.*

$\Delta[M,N] = \{\alpha : \alpha \in [M,N];\ Ker(\alpha) \leq_e M\}$

- *The co-singular ideal.*

$\nabla[M,N] = \{\alpha : \alpha \in [M,N];\ Im(\alpha) << M\}$

- *The total.*

$\mathrm{Tot}[M,N] = \{\alpha : \alpha \in [M,N]; [N,M]\alpha$ *contains no nonzero idempotent* $\}$

$\mathrm{Tot}[M,N] = \{\alpha : \alpha \in [M,N]; \alpha[N,M]$ *contains no nonzero idempotent* $\}$

Lemma 4.1. *Let* M_R, N_R *be modules then:*

(1) $\mathrm{Tot}[M,N] = \{\alpha : \alpha \in [M,N]; \beta\alpha \in \mathrm{Tot}(E_M)$ *for all* $\beta \in [N,M]\}$.

(2) $\mathrm{Tot}[M,N] = \{\alpha : \alpha \in [M,N]; \alpha\beta \in \mathrm{Tot}(E_M)$ *for all* $\beta \in [N,M]\}$.

Proof. (1). *Let* $\alpha \in \mathrm{Tot}[M,N]$. *suppose that* $\beta\alpha \notin \mathrm{Tot}(E_M)$ *for some* $\beta \in [N,M]$ *then there exists* $\gamma \in E_M$ *such that* $0 \neq \gamma(\beta\alpha) = [\gamma(\beta\alpha)]^2 \in E_M$. *Since* $\gamma\beta \in [N,M]$ *then* $0 \neq \gamma(\beta\alpha) = [(\gamma\beta)\alpha]^2 \in [N,M]\alpha$, *a contradiction. Let* $\alpha \in [M,N]$ *such that* $\beta\alpha \in \mathrm{Tot}(E_M)$ *for all* $\beta \in [N,M]\}$. *Suppose that* $\alpha \notin \mathrm{Tot}[\mathrm{M},\mathrm{N}]$ *then* $[N,M]\alpha$ *contains a nonzero idempotent. So* *there exists* $\gamma \in [N,M]$ *such that* $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$ *and* $\gamma\alpha \in (\gamma\alpha)E_M$, *so* $\gamma\alpha \notin \mathrm{Tot}(E_M)$, *is a contradiction. Similarly* (2) *holds.*

Lemma 4.2. *[14, Lemma 2.1]. Let* M_R, N_R *be modules. The following statements are equivalent:*

(1)*If* $\alpha \in [M,N] \setminus J[M,N]$ *there exists* $\beta \in [N,M]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$.

(2)*If* $\alpha \in [M,N] \setminus J[M,N]$ *there exists* $\beta \in [N,M]$ *such that* $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$.

(3) *If* $\alpha \in [M,N] \setminus J[M,N]$ *there exists* $\gamma \in [N,M]$ *such that* $\gamma\alpha\gamma = \gamma \notin J[N,M]$.

Following [14], recall that $[M,N]$ *is a semipotent if, the conditions in lemma 4.2, are satisfied.*

Lemma 4.3. *Let* M_R, N_R *be modules and* $[M,N]$ *is semipotent then:*

(1) $\Delta[M,N] \subseteq J[M,N]$ *and* $\nabla[M,N] \subseteq J[M,N]$.

(2) *If* $\Gamma(M) = \{0\}$ *then* $I[M,N] \subseteq J[M,N]$.

(3) *If* $\Gamma(N) = \{0\}$ *then* $I[M,N] \subseteq J[M,N]$.

Proof. Suppose that $[M,N]$ *is semipotent.*

(1). *Let* $\alpha \in \Delta[M,N]$ *then* $Ker(\alpha) \leq_e M$. *Suppose that* $\alpha \notin J[M,N]$ *then there exists* $\beta \in [N,M]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. *Since* $Ker(\alpha) \subseteq Ker(\beta\alpha)$ *then* $Ker(\alpha) \cap Im(\beta\alpha) \subseteq Ker(\beta\alpha) \cap Im(\beta\alpha) = 0$. *Thus,* $Im(\beta\alpha) = 0$ *and* $\beta\alpha = 0$ *this is a contradiction. Hence* $\alpha \in J[M,N]$. (2). *Suppose that* $\Gamma(M) = \{0\}$. *Let* $\alpha \in I[M,N]$ *then* $Im(\alpha) \subseteq J(N)$. *Assume that* $\alpha \notin J[M,N]$ *then there exists* $\beta \in [N,M]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. *So* $Im(\beta\alpha) \subseteq^{\oplus} M$ *and* $Im(\beta\alpha) \subseteq J(M)$ *thus* $Im(\beta\alpha) \in \Gamma(M) = \{0\}$, *so* $\beta\alpha = 0$ *is a contradiction. Thus* $\alpha \in J[M,N]$. *Similarly, (3) holds.*

Proposition 4.4. *Let* M_R, N_R *be modules, the following hold:*

(1) *If* $\text{Tot}(E_M) = J(E_M)$ *then* $\text{Tot}[M,N] = J[M,N]$.

(2) *If* $\text{Tot}(E_N) = J(E_N)$ *then* $\text{Tot}[M,N] = J[M,N]$.

(3) *If* E_M *is a semipotent ring then* $[M,N]$ *is a semipotent.*

(4) *If* E_N *is a semipotent ring then* $[M,N]$ *is a semipotent.*

Proof.(1). *Suppose that* $\text{Tot}(E_M) = J(E_M)$. *It is clear that* $J[M,N] \subseteq \text{Tot}[M,N]$. *Let* $\alpha \in \text{Tot}[M,N]$ *then by lemma 4.1 for any* $\beta \in [N,M]$, $\beta\alpha \in \text{Tot}(E_M) = J(E_M)$ *so* $\alpha \in J[M,N]$. *The proof of* (2) *is analogous.* (3) *Suppose that* E_M *is a semipotent ring then by [14, Theorem 2.2],* $\text{Tot}(E_M) = J(E_M)$ *and by* (1) $\text{Tot}[M,N] = J[M,N]$ *a gain by [14, Theorem 2.2],* $[M,N]$ *is semipotent. The proof of* (4) *is analogous.*

Remark. *Y. Zhou [14], gave an example of two modules* M_R, N_R *such that* $[M,N]$ *is semipotent, but neither* E_M, *nor* E_N *is semipotent, (See [14, Example 4.9]). So in general, if* $\text{Tot}[M,N] = J[M,N]$ *then* $\text{Tot}(E_M) \neq J(E_M)$ *and* $\text{Tot}(E_N) \neq J(E_N)$.

Following [14]. Let

$\Phi(R) = \{M \in mod-R : \text{Tot}[M,N] = J[M,N]$ *for all* $N \in mod-R\}$

$\Gamma(R) = \{N \in mod-R : \text{Tot}[M,N] = J[M,N]$ *for all* $M \in mod-R\}$

Corollary 4.5. *[14, Theorem 4.5]. The following hold:*

(1) $\Phi(R) = \{M \in mod-R : E_M$ *is a semipotent ring*$\}$.

(2) $\Gamma(R) = \{N \in mod-R : E_N$ *is a semipotent ring*$\}$.

(3) $\Phi(R) = \Gamma(R)$.

Proof. (1)($\Rightarrow$). *If* $M \in \Phi(R)$ *then* $\text{Tot}(E_M) = J(E_M)$, *so* E_M *is semipotent by [14، Theorem 2.2].* ($\Leftarrow$). *Let* $M \in mod-R$ *with* E_M *is a semipotent ring then for any* $N \in mod-R$; $[M,N]$ *is semipotent by proposition 4.4, so* $M \in \Phi(R)$. *Similarly* (2) *holds.* (3) *By* (1) *and* (2).

5. $(\Delta-, \nabla-, \text{I}-)$ Semipotent $[M,N]$

Proposition 5.1. *Let* M_R, N_R *be modules.*

(a) *The following hold:*

(1) $\Delta[M,N] \subseteq \{\alpha : \alpha \in [M,N]; \beta\alpha \in \Delta E_M$ *for all* $\beta \in [N,M]\}$.

(2) $\Delta[M,N] \subseteq \{\alpha : \alpha \in [M,N]; \alpha\beta \in \Delta E_N$ *for all* $\beta \in [N,M]\}$.

(b) *If* $\text{Tot}[M,N] = \Delta[M,N]$ *then*

(1) $\Delta[M,N] = \{\alpha : \alpha \in [M,N]; \beta\alpha \in \Delta E_M$ *for all* $\beta \in [N,M]\}$.

(2) $\Delta[M,N] = \{\alpha : \alpha \in [M,N]; \alpha\beta \in \Delta E_N$ *for all* $\beta \in [N,M]\}$.

Proof. (a). (1) *Let* $\alpha \in \Delta[M,N]$ *then* $Ker(\alpha) \leq_e M$ *, since for any* $\beta \in [N,M]$, $Ker(\alpha) \subseteq Ker(\beta\alpha)$ *then* $Ker(\beta\alpha) \leq_e M$ *thus* $\beta\alpha \in \Delta E_M$. (2) *Let* $\alpha \in \Delta[M,N]$ *then* $Ker(\alpha) \leq_e M$. *Let* $\beta \in [N,M]$ *and* K *be a submodule of* N *such that* $Ker(\alpha\beta) \cap K = 0$. *Since* $Ker(\beta) \subseteq Ker(\alpha\beta)$ *then* $Ker(\beta) \cap K = 0$. *Let* $y \in Ker(\alpha) \cap \beta(K)$ *then* $y \in Ker(\alpha)$, *so* $\alpha(y) = 0$ *and* $y \in \beta(K)$ *therefore* $y = \beta(x)$ *for some* $x \in K$. *So* $0 = \alpha(y) = \alpha\beta(x)$ *thus* $x \in Ker(\alpha\beta)$, $x \in K$, *so* $x \in Ker(\alpha\beta) \cap K = 0$ *thus* $x = 0$, *so* $y \in \beta(x) = 0$ *thus,* $Ker(\alpha) \cap \beta(K) = 0$. *Since* $Ker(\alpha) \leq_e M$ *follows that* $\beta(K) = 0$ *so* $K \subseteq Ker(\beta)$ *thus* $K = Ker(\beta) \cap K = 0$ *so* $Ker(\alpha\beta) \leq_e N$ *thus* $\alpha\beta \in \Delta E_N$.

(b). *Suppose that* $\text{Tot}[M,N] = \Delta[M,N]$. (1) *We have by* (a)

$\Delta[M,N] \subseteq \{\alpha : \alpha \in [M,N]; \beta\alpha \in \Delta E_M$ *for all* $\beta \in [N,M]\}$

Let $\alpha \in [M,N]$ *such that* $\beta\alpha \in \Delta E_M$ *for all* $\beta \in [N,M]$, *suppose* $\alpha \notin \Delta[M,N]$ *then* $\alpha \notin \text{Tot}[M,N]$ *so there exists* $\gamma \in [N,M]$ *such that* $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$ *therefore* $M = Im(\gamma\alpha) \oplus Ker(\gamma\alpha)$. *Since* $Ker(\gamma\alpha) \cap Im(\gamma\alpha) = 0$ *and* $Ker(\gamma\alpha) \leq_e M$ *follows* $Im(\gamma\alpha) = 0$, *so* $\gamma\alpha = 0$ *a contradiction. Thus,* $\alpha \in \Delta[M,N]$. *Similarly* (2) *holds.*

Lemma 5.2. *Let* M_R, N_R *be modules. The following statements are equivalent:*

(1) *If* $\alpha \in [M,N] \setminus \Delta[M,N]$ *there exists* $\beta \in [N,M]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$. (2) *If* $\alpha \in [M,N] \setminus \Delta[M,N]$ *there exists* $\beta \in [N,M]$ *such that* $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$.

(3) *If* $\alpha \in [M,N] \setminus \Delta[M,N]$ *there exists* $\gamma \in [N,M]$ *such that* $\gamma\alpha\gamma = \gamma \notin \Delta[N,M]$. *Proof. Suppose* (1) *holds. Then* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ *for some* $\beta \in [N,M]$. *Let* $\gamma = \beta\alpha\beta \in [N,M]$ *we have* $\gamma\alpha\gamma = \gamma \neq 0$ *and* $\gamma \notin \Delta[N,M]$ *because* $0 \neq \alpha\gamma = (\alpha\gamma)^2$, *giving* (3). *Suppose* (3) *holds, then* $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$ *for some* $\gamma \in [N,M] \setminus \Delta[N,M]$ *gives* (1). *Similarly, the equivalence* (2) $\Leftrightarrow$ (3) *holds.*

We call that $[M,N]$ *is* $\Delta-$*semipotent if the conditions in lemma 5.2 are satisfied.*

Theorem 5.3. *Let* M_R, N_R *be modules. Then* $[M,N]$ *is* $\Delta-$*semipotent if and only if,* $\mathrm{Tot}[M,N] = \Delta[M,N]$. *In particular,* E_M *is a* $\Delta-$*semipotent if and only if,* $\mathrm{Tot}(E_M) = \Delta E_M$.

Proof. $(\Rightarrow)$. *Suppose that* $\mathrm{Tot}[M,N] \neq \Delta[M,N]$. *Since* $\Delta[M,N] \subset \mathrm{Tot}[M,N]$, *there exists* $\alpha \in \mathrm{Tot}[M,N]$ *such that* $\alpha \notin \Delta[M,N]$. *So, for any* $\beta \in [N,M]$ *either* $\alpha\beta = 0$ *or* $\alpha\beta \neq (\alpha\beta)^2$. *Hence* $[M,N]$ *is not* $\Delta-$*semipotent.* $(\Leftarrow)$. *If* $\alpha \in [M,N] \setminus \Delta[M,N]$ *then* $\alpha \notin \mathrm{Tot}[M,N]$. *So* $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$ *for some* $\beta \in [N,M]$. *This shows that* $[M,N]$ *is* $\Delta-$ *semipotent.*

Let

$\Delta\Phi(R) = \{M \in mod-R : \mathrm{Tot}[M,N] = \Delta[M,N]$ *for all* $N \in mod-R\}$

$\Delta\Gamma(R) = \{N \in mod-R : \mathrm{Tot}[M,N] = \Delta[M,N]$ *for all* $M \in mod-R\}$

We define the following two sets:

(a) $\Delta S\,\Phi(R)$ *the set of all modules* $M \in mod-R$ *which satisfies the following two properties:*

(1) E_M *is a* $\Delta-$ *semipotent ring.*

(2) *For any* $N \in mod-R$;

$\Delta[M,N]=\{\alpha:\alpha\in[M,N];\ \beta\alpha\in\Delta E_M$ *for all* $\beta\in[N,M]\}$.

(b) $\Delta S\Gamma(R)$ *the set of all modules* $N\in mod-R$ *which satisfies the following two properties:*

(1) E_N *is a* $\Delta-$ *semipotent ring.*

(2) *For any* $M\in mod-R$*;*

$\Delta[M,N]=\{\alpha:\alpha\in[M,N];\ \alpha\beta\in\Delta E_N$ *for all* $\beta\in[N,M]\}$.

Theorem 5.4. *The following hold:*

(1) $\Delta\Phi(R)=\Delta S\Phi(R)$.

(2) $\Delta\Gamma(R)=\Delta S\Gamma(R)$.

(3) $\Delta\Phi(R)=\Delta\Gamma(R)$.

Proof. $(1)(\Rightarrow)$. *Let* $M\in\Delta\Phi(R)$ *then for any* $N\in mod-R$*;* $\mathrm{Tot}[M,N]=\Delta[M,N]$ *by proposition 5.1*(b). *We have* $\Delta[M,N]=\{\alpha:\alpha\in[M,N];\ \beta\alpha\in\Delta E_M$ *for all* $\beta\in[N,M]\}$ *and it is clear that* E_M *is a* $\Delta-$*semipotent ring, so* $M\in\Delta S\Phi(R)$.

$(\Leftarrow)$. *Let* $M\in\Delta S\Phi(R)$. *We have for any* $N\in mod-R$, $\Delta[M,N]\subseteq\mathrm{Tot}[M,N]$. *Let* $\alpha\in\mathrm{Tot}[M,N]$ *then by lemma 4.1 for any* $\beta\in[N,M]$*;* $\beta\alpha\in\mathrm{Tot}(E_M)$. *Since* E_M *is* $\Delta-$ *semipotent then by theorem 5.3,* $\mathrm{Tot}(E_M)=\Delta E_M$*, so* $\beta\alpha\in\Delta E_M$ *for all* $\beta\in[N,M]$ *thus,* $M\in\Delta\Phi(R)$.

$(2)(\Rightarrow)$. *Let* $N\in\Delta\Gamma(R)$ *then for any* $M\in mod-R$*;* $\mathrm{Tot}[M,N]=\Delta[M,N]$ *by proposition 5.1*(b). *we have* $\Delta[M,N]=\{\alpha:\alpha\in[M,N];\ \alpha\beta\in\Delta E_N$ *for all* $\beta\in[N,M]\}$ *and* E_N *is a* $\Delta-$ *semipotent ring, so* $N\in\Delta S\Gamma(R)$.

$(\Leftarrow)$. *Let* $N\in\Delta S\Gamma(R)$ *then for any* $M\in mod-R$ *we have* $\Delta[M,N]\subseteq\mathrm{Tot}[M,N]$. *Let* $\alpha\in\mathrm{Tot}[M,N]$ *by lemma 4.1 for any* $\beta\in[N,M]$*;* $\alpha\beta\in\mathrm{Tot}(E_N)$. *Since* E_N *is a* $\Delta-$*semipotent ring then by theorem 5.3,* $\mathrm{Tot}(E_N)=\Delta E_N$ *so* $\alpha\beta\in\Delta E_N$ *for all* $\beta\in[N,M]$ *by assumption* $\alpha\in\Delta[M,N]$. *Thus,* $N\in\Delta\Phi(R)$. (3). *By* (1) *and* (2).

Proposition 5.5. *Let* M_R, N_R *be modules.*

(a) *The following hold:*

(1) $\nabla[M,N]\subseteq\{\alpha:\alpha\in[M,N];\ \beta\alpha\in\nabla E_M$ *for all* $\beta\in[N,M]\}$.

(2) $\nabla[M,N]\subseteq\{\alpha:\alpha\in[M,N];\ \alpha\beta\in\nabla E_N$ *for all* $\beta\in[N,M]\}$.

(b) *If* $\mathrm{Tot}[M,N]=\nabla[M,N]$ *then*

(1) $\nabla[M,N]=\{\alpha:\alpha\in[M,N];\ \beta\alpha\in\nabla E_M$ *for all* $\beta\in[N,M]\}$.

(2) $\nabla[M,N]=\{\alpha:\alpha\in[M,N];\ \alpha\beta\in\nabla E_N$ *for all* $\beta\in[N,M]\}$.

Proof. (a). (1) *Let* $\alpha\in\nabla[M,N]$ *then* $Im(\alpha)<<N$, *so for any* $\beta\in[N,M]$, $Im(\beta\alpha)<<M$ *thus* $\beta\alpha\in\nabla E_M$.

(2) *Let* $\alpha\in\nabla[M,N]$ *then* $Im(\alpha)<<N$, *since for all* $\beta\in[N,M]$ $Im(\alpha\beta)\subseteq Im(\alpha)$

then $Im(\alpha\beta)<<N$ *so* $\alpha\beta\in\nabla E_N$.

(b). *Suppose that* $\mathrm{Tot}[M,N]=\nabla[M,N]$. (1) *We have by* (a)

$\nabla[M,N]\subseteq\{\alpha:\alpha\in[M,N];\ \beta\alpha\in\nabla E_M$ *for all* $\beta\in[N,M]\}$.

Let $\alpha\in[M,N]$ *such that* $\beta\alpha\in\nabla E_M$ *for all* $\beta\in[N,M]$, *suppose* $\alpha\notin\nabla[M,N]$ *then* $\alpha\notin\mathrm{Tot}[M,N]$ *so there exists* $\gamma\in[N,M]$ *such that* $0\neq\gamma\alpha=(\gamma\alpha)^2\in E_M$ *therefore* $0\neq Im(\gamma\alpha)\subseteq^{\oplus}M$. *Since* $\gamma\alpha\in\nabla E_M$; $Im(\gamma\alpha)<<M$ *so* $Im(\gamma\alpha)=0$ *a contradiction. Thus,* $\alpha\in\nabla[M,N]$. *Similarly* (2) *holds.*

Lemma 5.6. *Let* M_R, N_R *be modules. The following are equivalent:*

(1) *If* $\alpha\in[M,N]\setminus\nabla[M,N]$ *there exists* $\beta\in[N,M]$ *such that* $0\neq\beta\alpha=(\beta\alpha)^2\in E_M$. (2) *If* $\alpha\in[M,N]\setminus\nabla[M,N]$ *there exists* $\beta\in[N,M]$ *such that* $0\neq\alpha\beta=(\alpha\beta)^2\in E_N$. (3) *If* $\alpha\in[M,N]\setminus\nabla[M,N]$ *there exists* $\gamma\in[N,M]$ *such that* $\gamma\alpha\gamma=\gamma\notin\nabla[N,M]$. *Proof.* $(1)\Rightarrow(3)$. *Let* $\alpha\in[M,N]\setminus\nabla[M,N]$, *then* $0\neq\beta\alpha=(\beta\alpha)^2\in E_M$ *for some* $\beta\in[N,M]$. *Let* $\gamma=\beta\alpha\beta$ *then* $\gamma\in[N,M]$ *and* $\gamma\alpha\gamma=\gamma\notin\nabla[N,M]$ *because* $\beta\alpha\notin\nabla E_M$. *Suppose* (3) *holds، let* $\alpha\in[M,N]\setminus\nabla[M,N]$ *then* $\gamma=\gamma\alpha\gamma$ *for some* $\gamma\in[N,M]\setminus\nabla[N,M]$ *so* $0\neq\gamma\alpha=(\gamma\alpha)^2\in E_M$, *gives* (1). *Similarly, the equivalence* $(2)\Leftrightarrow(3)$ *holds.*

We call that $[M,N]$ *is* $\nabla-$*semipotent if the conditions in lemma 5.6 are satisfied.*

Theorem 5.7. *Let* M_R, N_R *be modules.* $[M,N]$ *is* $\nabla-$ *semipotent if and only if,* $\text{Tot}[M,N]=\nabla[M,N]$. *In particular,* E_M *is a* $\nabla-$ *semipotent if and only if,* $\text{Tot}(E_M)=\nabla E_M$.

Proof. $(\Rightarrow)$. *Suppose that* $\text{Tot}[M,N]\neq\nabla[M,N]$, *Since* $\nabla[M,N]\subset\text{Tot}[M,N]$, *there exists* $\alpha\in\text{Tot}[M,N]$ *such that* $\alpha\notin\nabla[M,N]$. *So, for any* $\beta\in[N,M]$ *either* $\alpha\beta\neq(\alpha\beta)^2$ *or* $\alpha\beta=0$. *Hence* $[M,N]$ *is not* $\nabla-$*semipotent.* $(\Leftarrow)$. *If* $\alpha\in[M,N]\setminus\nabla[M,N]$ *then* $\alpha\notin\text{Tot}[M,N]$. *So* $0\neq\alpha\beta=(\alpha\beta)^2\in E_N$ *for some* $\beta\in[N,M]$. *This shows that* $[M,N]$ *is* $\nabla-$ *semipotent.*

Let

$\nabla\Phi(R)=\{M\in mod-R:\text{Tot}[M,N]=\nabla[M,N]$ *for all* $N\in mod-R\}$

$\nabla\Gamma(R)=\{N\in mod-R:\text{Tot}[M,N]=\nabla[M,N]$ *for all* $M\in mod-R\}$

We define the following two sets:

(a) $\nabla S\,\Phi(R)$ *the set of all modules* $M\in mod-R$ *which satisfies the following two properties:*

(1) E_M *is a* $\nabla-$ *semipotent ring.*

(2) *For any* $N\in mod-R$; $\nabla[M,N]=\{\alpha:\alpha\in[M,N];\beta\alpha\in\nabla E_M$ *for all* $\beta\in[N,M]\}$.

(b) $\nabla S\,\Gamma(R)$ *the set of all modules* $N\in mod-R$ *which satisfies the following two properties:*

(1) E_N *is a* $\nabla-$ *semipotent ring.*

(2) *For any* $M\in mod-R$;

$\nabla[M,N]=\{\alpha:\alpha\in[M,N];\alpha\beta\in\nabla E_N$ *for all* $\beta\in[N,M]\}$.

Theorem 5.8. *The following hold:*

(1) $\nabla\Phi(R)=\nabla S\,\Phi(R)$.

(2) $\nabla\Gamma(R)=\nabla S\,\Gamma(R)$.

(3) $\nabla\Phi(R)=\nabla\Gamma(R)$.

Proof. $(1)(\Rightarrow)$. *Let* $M \in \nabla\Phi(R)$ *then for any* $N \in mod-R$; $\mathrm{Tot}[M,N] = \nabla[M,N]$ *by proposition 5.5(b) we have* $\nabla[M,N] = \{\alpha : \alpha \in [M,N]; \beta\alpha \in \nabla E_M$ *for all* $\beta \in [N,M]\}$. *In addition to,* E_M *is a* $\nabla-$*semipotent ring, so* $M \in \nabla S\,\Phi(R)$. $(\Leftarrow)$. *Let* $M \in \nabla S\,\Phi(R)$. *We have for any* $N \in mod-R$, $\nabla[M,N] \subseteq \mathrm{Tot}[M,N]$. *Let* $\alpha \in \mathrm{Tot}[M,N]$ *then by lemma 4.1 for any* $\beta \in [N,M]$; $\beta\alpha \in \mathrm{Tot}(E_M)$. *Since* E_M *is* $\nabla-$*semipotent then by theorem 5.7* $\mathrm{Tot}(E_M) = \nabla E_M$, *so* $\beta\alpha \in \nabla E_M$ *for all* $\beta \in [N,M]$ *thus,* $M \in \nabla\Phi(R)$.

$(2)(\Rightarrow)$. *Let* $N \in \nabla\Gamma(R)$ *then for any* $M \in mod-R$; $\mathrm{Tot}[M,N] = \nabla[M,N]$ *by proposition 5.5(b) we have* $\nabla[M,N] = \{\alpha : \alpha \in [M,N]; \beta\alpha \in \nabla E_N$ *for all* $\beta \in [N,M]\}$. *In addition to,* E_N *is a* $\nabla-$ *semipotent ring, so* $N \in \nabla S\,\Gamma(R)$.

$(\Leftarrow)$.*Let* $N \in \nabla S\,\Gamma(R)$ *then for any* $M \in mod-R$ *we have* $\nabla[M,N] \subseteq \mathrm{Tot}[M,N]$. *Let* $\alpha \in \mathrm{Tot}[M,N]$ *by lemma 4.1 for any* $\beta \in [N,M]$; $\alpha\beta \in \mathrm{Tot}(E_N)$. *Since* E_N *is a* $\nabla-$*semipotent ring then by theorem 5.7,* $\mathrm{Tot}(E_N) = \nabla E_N$ *so* $\alpha\beta \in \nabla E_N$ *for all* $\beta \in [N,M]$ *by assumption* $\alpha \in \nabla[M,N]$. *Thus,* $N \in \nabla\Phi(R)$. (3). *By* (1) *and* (2).

Let M_R, N_R *be modules. We put* $I[M,N] = \{\alpha : \alpha \in [M,N]; Im(\alpha) \subseteq J(N)\}$. *Since any small submodule of* N *contained in* $J(N)$ *then* $\nabla[M,N] \subseteq I[M,N]$. *If* $J(N) << N$ *then* $I[M,N] = \nabla[M,N]$. *Thus*

$I = I(E_M) = I[M,M] = \{\alpha : \alpha \in E_M; Im(\alpha) \subseteq J(M)\}$.

In particular for a ring R, $I(R) = I[R,R] = J[R,R] = J(R)$. *A gain for a module* M_R *we put* $\Gamma(M) = \{K; K \subseteq^{\oplus} M$ *and* $K \subseteq J(M)\}$.

Proposition 5.9. *Let* M_R, N_R *be modules.*

(a) *The following hold:*

(1) $I[M,N] \subseteq \{\alpha : \alpha \in [M,N]; \beta\alpha \in I(E_M)$ *for all* $\beta \in [N,M]\}$.

(2) $I[M,N] \subseteq \{\alpha : \alpha \in [M,N]; \alpha\beta \in I(E_N)$ *for all* $\beta \in [N,M]\}$.

(b) *If* $\mathrm{Tot}[M,N] = I[M,N]$ *and* $\Gamma(M) = \{0\}$ *then*

$I[M,N] = \{\alpha : \alpha \in [M,N]; \beta\alpha \in I(E_M)$ *for all* $\beta \in [N,M]\}$.

(c) *If* $\mathrm{Tot}[M,N] = I[M,N]$ *and* $\Gamma(N) = \{0\}$ *then*

$I[M,N] = \{\alpha : \alpha \in [M,N]; \alpha\beta \in I(E_N)$ *for all* $\beta \in [N,M]\}$.

Proof.(a)(1). *Let* $\alpha \in I[M,N]$ *then* $Im(\alpha) \subseteq J(N)$, *so for any* $\beta \in [N,M]$; $\beta\alpha \in E_M$ *and* $Im(\beta\alpha) \subseteq J(M)$. *Thus,* $\beta\alpha \in I(E_M)$.

(2) *Let* $\alpha \in I[M,N]$ *then* $Im(\alpha) \subseteq J(N)$, *so for any* $\beta \in [N,M]$; $\alpha\beta \in E_N$ *and* $Im(\alpha\beta) \subseteq J(N)$. *Thus*، $\alpha\beta \in I(E_N)$.

(b). *Suppose that* $\mathrm{Tot}[M,N] = I[M,N]$ *and* $\Gamma(M) = \{0\}$. *Let* $\alpha \in [M,N]$ *such that* $\beta\alpha \in I(E_M)$ *for all* $\beta \in [N,M]$. *Suppose* $\alpha \notin I[M,N]$ *then* $\alpha \notin \mathrm{Tot}[M,N]$, *so there exists* $\beta \in [N,M]$ *such that* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$, *since* $Im(\beta\alpha) \subseteq J(M)$ *and* $Im(\beta\alpha) \subseteq^{\oplus} M$ *then* $Im(\beta\alpha) \in \Gamma(M) = \{0\}$, *a contradiction.*

(c). *Suppose that* $\mathrm{Tot}[M,N] = I[M,N]$ *and* $\Gamma(N) = \{0\}$. *Let* $\alpha \in [M,N]$ *such that* $\alpha\beta \in I(E_N)$ *for all* $\beta \in [N,M]$. *Suppose* $\alpha \notin I[M,N]$ *then* $\alpha \notin \mathrm{Tot}[M,N]$, *so there exists* $\gamma \in [N,M]$ *such that* $0 \neq \alpha\gamma = (\alpha\gamma)^2 \in E_N$. *Since* $Im(\alpha\gamma) \subseteq J(N)$ *and* $Im(\alpha\gamma) \subseteq^{\oplus} N$ *then* $Im(\alpha\gamma) \in \Gamma(N) = \{0\}$, *a contradiction. Thus* $\alpha \in I[M,N]$.

Lemma 5.10. *Let* M_R, N_R *be modules. The following statements are equivalent:*

(1) *If* $\alpha \in [M,N] \setminus I[M,N]$, *there exists* $\beta \in [N,M]$; $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ *and* $\beta\alpha \notin I(E_M)$.

(2) *If* $\alpha \in [M,N] \setminus I[M,N]$, *there exists* $\beta \in [N,M]$; $0 \neq \alpha\beta = (\alpha\beta)^2 \in E_N$ *and* $\alpha\beta \notin I(E_N)$.

(3) *If* $\alpha \in [M,N] \setminus I[M,N]$, *there exists* $\gamma \in [N,M]$; $\gamma\alpha\gamma = \gamma \notin I[N,M]$.

Proof. Suppose (1) *holds. Then* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ *and* $\beta\alpha \notin I(E_M)$ *for some* $\beta \in [N,M]$. *By letting* $\gamma = \beta\alpha\beta \in [N,M]$ *we* *have* $\gamma\alpha\gamma = \gamma \neq 0$ *and* $\gamma \notin I[N,M]$ *because* $\beta\alpha \notin I(E_M)$, *giving* (3). *Suppose* (3) *holds. Then* $0 \neq \gamma\alpha = (\gamma\alpha)^2 \in E_M$ *and* $\gamma\alpha \notin I(E_M)$ *because* $\gamma \notin I[N,M]$ *gives*(1). *Similarly, the equivalence* $(2) \Leftrightarrow (3)$ *holds.*

We call that $[M,N]$ *is* $I-$*semipotent if the conditions in lemma 5.10 are satisfied.*

Theorem 5.11. *Let* M_R, N_R *be modules then the following hold:*

(1) *If* $\Gamma(M) = \{0\}$ *then* $\mathrm{Tot}[M,N] = I[M,N]$ *if and only if,* $[M,N]$ *is* $I-$*semipotent.*

(2) *If* $\Gamma(N) = \{0\}$ *then* $\mathrm{Tot}[M,N] = I[M,N]$ *if and only if,* $[M,N]$ *is* $I-$*semipotent.*

In particular, if $\Gamma(M) = \{0\}$ *then* $\mathrm{Tot}(E_M) = I(E_M)$ *if and only if,* E_M *is an* $I-$ *semipotent ring.*

Proof. (1). *Suppose that* $\Gamma(M) = \{0\}$. $(\Rightarrow)$ *let* $\alpha \in [M,N] \setminus I[M,N]$ *then* $\alpha \notin \mathrm{Tot}[M,N]$, *so* $0 \neq \beta\alpha = (\beta\alpha)^2 \in E_M$ *for some* $\beta \in [N,M]$ *and* $\beta\alpha \notin I(E_M)$ *because* $\Gamma(M) = \{0\}$. *This shows that* $[M,N]$ *is I-semipotent.* $(\Leftarrow)$ *Since* $\Gamma(M) = \{0\}$ *it is easy to see that* $I[M,N] \subseteq \mathrm{Tot}[M,N]$. *Let* $\alpha \in \mathrm{Tot}[M,N]$ *and suppose* $\alpha \notin I[M,N]$ *so, for any* $\beta \in [N,M]$, *either* $\alpha\beta = 0$ *or* $\alpha\beta \neq (\alpha\beta)^2$. *Hence* $[M,N]$ *is not* $I-$ *semipotent. Similarly* (2) *holds.*

Let

$I\Phi(R) = \{M \in mod-R : \Gamma(M) = \{0\}$ *and* $\mathrm{Tot}[M,N] = I[M,N]; \forall\ N \in mod-R\}$

$I\Gamma(R) = \{N \in mod-R : \Gamma(N) = \{0\}$ *and* $\mathrm{Tot}[M,N] = I[M,N]; \forall\ M \in mod-R\}$

We define the following two sets:

(a) $IS\Phi(R)$ *the set of all modules* $M \in mod-R$ *which satisfies the following properties:*

(1) $\Gamma(M) = \{0\}$.

(2) E_M *is an* $I-$*semipotent ring.*

(3) *For any* $N \in mod-R$;

$I[M,N]=\{\alpha:\alpha\in[M,N];\ \beta\alpha\in I(E_M)\ \text{for all}\ \beta\in[N,M]\}$.

(b) $IS\Gamma(R)$ *the set of all modules* $N \in mod-R$ *which satisfies the following properties:*

(1) $\Gamma(N)=\{0\}$.

(2) E_N *is an* $I-$*semipotent ring.*

(3) *For any* $M \in mod-R$;

$I[M,N]=\{\alpha:\alpha\in[M,N];\ \alpha\beta\in I(E_N)\ \text{for all}\ \beta\in[N,M]\}$.

Theorem 5.12. *The following hold:*

(1) $I\Phi(R)=IS\Phi(R)$.

(2) $I\Gamma(R)=IS\Gamma(R)$.

(3) $I\Phi(R)=I\Gamma(R)$.

Proof. $(1)(\Rightarrow)$. *Let* $M \in I\Phi(R)$ *then* $\Gamma(M)=\{0\}$ *and* $\mathrm{Tot}[M,N]=I[M,N]$ *for all* $N\in mod-R$. *So,* $\mathrm{Tot}(E_M)=I(E_M)$ *and by theorem 5.11،* E_M *is an* $I-$*semipotent ring. On the other hand, by proposition 5.9(b) we have for any* $N\in mod-R$; $I[M,N]=\{\alpha:\alpha\in[M,N];\ \beta\alpha\in I(E_M)\ \text{for all}\ \beta\in[N,M]\}$. *So,* $M\in IS\Phi(R)$.

$(\Leftarrow)$. *Let* $M\in IS\Phi(R)$ *then* $\Gamma(M)=\{0\}$. *Let* $N\in mod-R$ *and* $\alpha\in I[M,N]$ *then* $Im(\alpha)\subseteq J(N)$. *Suppose that* $\alpha\notin\mathrm{Tot}[M,N]$ *then there exists* $\beta\in[N,M]$ *such that* $0\neq\beta\alpha=(\beta\alpha)^2\in E_M$. *So,* $0\neq Im(\beta\alpha)\subseteq^{\oplus} M$ *and* $Im(\beta\alpha)\in\Gamma(M)=\{0\}$, *a contr-adiction. Thus,* $I[M,N]\subseteq\mathrm{Tot}[M,N]$. *Let* $\alpha\in\mathrm{Tot}[M,N]$ *and suppose that* $\alpha\notin I[M,N]$, *since* $M\in IS\Phi(R)$ *then there exists* $\beta\in[N,M]$ *such that* $\beta\alpha\notin I(E_M)$. *Since* E_M *is an* $I-$*semipotent ring there exists* $\gamma\in E_M$ *such that* $\gamma(\beta\alpha)\gamma=\gamma\notin I(E_M)$ *thus,* $0\neq(\gamma\beta)\alpha=[(\gamma\beta)\alpha]^2\in E_M$ *and* $\gamma\beta\in[N,M]$, *a contradiction, hence* $\alpha\in\mathrm{Tot}[M,N]$, *therefore* $\alpha\in I[M,N]$. *Thus،* $\mathrm{Tot}[M,N]=I[M,N]$ *for any* $N\in mod-R$, *so* $M\in I\Phi(R)$.

$(2)(\Rightarrow)$. *Let* $N \in I\Gamma(R)$ *then* $\Gamma(N)=\{0\}$ *and* $\text{Tot}[M,N]=I[M,N]$ *for all* $M \in mod-R$. *So,* $\text{Tot}(E_N)=I(E_N)$ *and by theorem 5.11,* E_N *is an* $I-$*semipotent ring. On the other hand, by proposition 5.9*(c) *we have for any* $M \in mod-R$*;* $I[M,N]=\{\alpha:\alpha\in[M,N];\alpha\beta\in I(E_N)$ *for all* $\beta\in[N,M]\}$. *So،* $N \in IS\Gamma(R)$. *Let* $N\in IS\Gamma(R)$ *then* $\Gamma(N)=\{0\}$. *Let* $M\in mod-R$ *and* $\alpha\in I[M,N]$ *then* $Im(\alpha)\subseteq J(N)$. *Suppose that* $\alpha\notin\text{Tot}[M,N]$ *then there exists* $\beta\in[N,M]$ *such that* $0\neq\alpha\beta=(\alpha\beta)^2\in E_N$. *So,* $0\neq Im(\alpha\beta)\subseteq^{\oplus} N$ *and* $Im(\alpha\beta)\in\ \Gamma(N)=\{0\}$, *a cont-radiction. Thus,* $I[M,N]\subseteq\text{Tot}[M,N]$. *Let* $\alpha\in\text{Tot}[M,N]$ *and suppose that* $\alpha\notin I[M,N]$, *since* $N\in IS\Gamma(R)$ *then there exists* $\beta\in[N,M]$ *such that* $\alpha\beta\notin\ I(E_N)$. *Since* E_N *is an* $I-$ *semipotent there exists* $\gamma\in E_N$ *such that* $\gamma(\alpha\beta)\gamma=\gamma\notin\ I(E_N)$ *thus,* $0\neq(\alpha\beta)\gamma=[(\alpha\beta)\gamma]^2\in E_N$ *and* $\beta\gamma\in[N,M]$, *a contradiction, hence* $\alpha\in\text{Tot}[M,N]$, *therefore* $\alpha\in I[M,N]$. *Thus،* $\text{Tot}[M,N]=I[M,N]$ *for any* $M\in mod-R$, *so* $N\in I\Gamma(R)$. (3). *By* (1) *and* (2).

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