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Generalized Right Bear Rings

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ABSTRACT

The object of this paper is to study the relationship between certain ring R and endomorphism rings of free modules over *R*. Specifically, the basic problem is to describe ring *R*, which is endomorphism ring of all free R-module, as a generalized right Bear ring. Call a ring R a generalized right Bear ring if any right annihilator contains a nonzero idempotent. A structure theorem is obtained: endomorphism ring of a free module F is a generalized right Bear ring if and only if every closed submodule of F contains a direct summand of *F*. It is shown that every torsionless R-module contains a projective R-module if endomorphism ring of any free R-module is a generalized right Bear ring.

Key words: Projective module, Torsinoless module, Closed module, Bear ring, Endomorphism ring.

حلقـات بيـر المعممـة

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المـلخص

موضوع هذا البحث هو دراسة العلاقة بين حلقة ما R وبين حلقة الإندومورفيزمات لمـودول حـر F، فوق هذه الحلقة. وبشكل خاص فقد تم وصف الحلقة التي تكون لأجلها حلقة الإندومورفيزمات لأي مودول حر فوقها، وهي حلقة بير المعممة. حيث تم إثبـات أن الـشرط الـلازم والكـافي لكـي تكـون حلقـة الإندومورفيزمات لمودول حر حلقة بير هو أن يحوي كل مودول جزئي مغلق من هذا المودول الحر حـداً مباشراً.

كذلك فقد تم إثبات أنه إذا كانت حلقة الاندومورفيزمات لمودول حر F حلقة بير المعممة, فإن كـل مودول جزئي دون فتل من F يحوي مودولاً إسقاطياً.

الكلمات المفتاحية: المودول الإسقاطي، المودول دون فتل، المودول المغلق، حلقة بير، حلقة التشاكلات لمودول.

Throughout this paper R *means an associative ring with identity element, and modules mean unitary* R*-modules.*

For any non-empty subset A *of a ring* R*, we denote the right annihilator of* A *in* R *by* $r(A) = \{a : a \in R; Aa = 0\}$*. Similarly, the left annihilator of* A *in* R *denoted by* $l(A) = \{a : a \in R; aA = 0\}$*. A right ideal of the form* $r(X)$ *for some non-empty subset* X *of* R*, is called a right annihilator. A left annihilator is defined similarly.*

Now, we begin with the following lemma.

Lemma 1. *For any ring* R *, the following statements are equivalent:*

(1)- *For any non-empty subset* A *of* R *there exists an idempotent* $1 \neq f \in R$ *(resp.* $0 \neq f \in R$*) such that* $l(A) \subseteq Rf$ *(resp.* $f \in l(A)$*).*

(2)- *For any non-empty subset* B *of* R *there exists an idempotent* $0 \neq e \in R$ *(resp.* $1 \neq e \in R$*) such that* $e \in r(B)$ *(resp.* $r(B) \subseteq eR$*).*

(3)- *For any non-empty subset* D *of* R *there exists an idempotent* $1 \neq g \in R$ *such that* $a = ag$ *(resp.* $a = ga$*) for all* $a \in D$.

Proof. (1)$\Rightarrow$ (2). *Let* B *be a non-empty subset of* R *and* $b \in r(B)$. *Then* $yb = 0$ *and* $y \in l(b)$ *for all* $y \in B$. *By (1) there exists an idempotent* $1 \neq f \in R$ *such that* $l(b) \subseteq Rf$. *Thus* $y(1-b) = 0$ *for all* $y \in B$. *Therefore* $(1-f) \in r(B)$ *and 1-f is a nonzero idempotent of* R.

(2)$\Rightarrow$ (3). *Let* D *be a non-empty subset of* R*, by (2),* $r(D)$ *contains a nonzero idempotent* e*, therefore* $De = 0$ *and* $a = a(1-e)$ *for all* $a \in D$*, where* $1 \neq 1-e \in R$ *is an idempotent.*

(3)$\Rightarrow$ (1). *It is clear.*

Definition. *We call a ring R a generalized right (resp. left) Baer ring if it satisfies the equivalent conditions of lemma 1. A ring, which is both generalized right and left, is called generalized Baer ring.*

Following [1], we call a ring R *Baer ring if every annihilator right ideal of R is generated by an idempotent, or equivalently, every annihilator left ideal of R is generated by an idempotent. It is clear every Baer ring is a generalized Baer ring.*

Lemma 2. *If* R *is a generalized right (resp. left) Baer ring, so is the ring eRe for all idempotent* e *of* R.

Proof. Let A *be a non-empty subset of eRe, write for* $l_e(A)$ *the left annihilator of* A *in eRe. Note that* $l_e(A) = eRe \cap l(A)$. *Since,* $A \subseteq R$ *and* R *is a generalized right Baer ring, there exists an idempotent* $f \neq 1$ *in* R *such that* $l(A) \subseteq Rf$ *.On the other hand,* $(1-e) \in l(A)$, *therefore* $(1-e) \in Rf$ *and* $(1-e) = rf = rff = (1-e)f$ *for some* $r \in R$. *So that* $f = ef + (1-e)$ *from which follows* $fe = efe$. *Thus* $g = fe \neq 1$ *is an idempotent in eRe. Suppose* $x \in l_e(A) = l(A) \cap eRe$, *then* $x \in l(A) \subseteq Rf$ *and* $x = xf$, $x = xe = xfe = xg \in eReg$. *Therefore* $l_e(A) \subseteq eReg$.

For any R*-module* M , M^* *denotes* $Hom_R(M,R)$ *and* S *denotes* $End_R(M)$. *Also for any submodule* A *of* M , *and for any non-empty subset* I *of* S , *we denote*

$$K(A) = \{y : y \in M; Iy = 0\},\ N(A) = \{f : f \in S; f(A) = 0\},$$
$$Q(A) = \{\psi : \psi \in S; \psi(M) \subseteq A\},\ A' = \{\mu : \mu \in M^*; \mu(A) = 0\},$$
$$A'' = \{y : y \in M; A'y = 0\}$$

It is easy to see that $K(A)$ *is a submodule of* M , $N(A)$ *is a left ideal in* S *and* $A \subseteq A'', A''$ *is a submodule of* M .

Following [2], let A *be a submodule of* R*-module* M , *we call a submodule* A'' *a closer to* A. *And* A *is called a closed submodule if* $A = A''$. *We will need some results from paper [2] which are mentioned throughout the following lemma.*

Lemma 3. *If* A *is a submodule of a free* R*-module* F *then:*

(1)- $A'' = K(N(A)) = \bigcap_{x_\alpha \in N(A)} Ker x_\alpha$ *[1, lemma 4].*

(2)- *Let* $I = \{x_\alpha\}$ *be a subset of* S*, then*

- *If* $A = \sum_{x_\alpha \in I} \operatorname{Im} x_\alpha$, *then* $l(I) = N(A)$ *[1, lemma 5].*
- *If* $D = \bigcap_{x_\alpha \in I} Ker x_\alpha$, *then* $r(I) = Q(D)$ *[1,lemma 6].*

(3)- $K(N(A)) = A$ *if and only if* $A = \bigcap_{\sigma_\alpha \in \Im} Ker \sigma_\alpha$ *for some subset* $\Im$ *in* S. *[1, corollary 3].*

Proposition 4. *Let* F *be a free* R*-module and* $I = \{x_\alpha\}$ *be a subset of* $S = End_R(F)$*, then the following conditions are equivalent:*

(1)- $l(I) \subseteq Se$ *for some idempotent* e *of* S.

(2)- *The closer to* $A = \sum_{x_\alpha \in I} \operatorname{Im} x_\alpha$ *contains a direct summand of* F.

Proof. (1)$\Rightarrow$(2). *Assume* $l(I) \subseteq Se$ *for some idempotent* e *of* S. *By lemma 3(2) we have* $l(I) = N(A)$ *where* $A = \sum_{x_\alpha \in I} \operatorname{Im} x_\alpha$. *Therefore by lemma 3(1)*

$$A'' = K(N(A)) = \bigcap_{\sigma_\alpha \in N(A)} Ker\, \sigma_\alpha = \bigcap_{\sigma_\alpha \in l(I)} Ker \sigma_\alpha .$$

Let $\sigma_\alpha \in l(I)$, *then* $\sigma_\alpha = \lambda e$ *for some* $\lambda \in S$ *and* $Ker e \subseteq Ker \sigma_\alpha$ *for all* $\sigma_\alpha \in l(I)$. *This shows that* $Ker e \subseteq \bigcap_{\sigma_\alpha \in l(I)} Ker \sigma_\alpha = A''$, *where* $Ker\, e$ *is a direct summand of* F.

$(2)\Rightarrow(1)$. *Suppose that* F_0 *is a direct summand of* F *, and* $F_0 \subseteq A''$. *Then* $F = F_0 \oplus P$ *where* P *a submodule of* F. *Let* e *be the projection of* F *onto* P*, then it is easy to see that* e *is an idempotent of* S *and* $e(x) = x$ *for all* $x \in P$. *Let* $g \in l(I)$ *by lemma 3* $g \in l(I) = N(A) = N(A'')$, *and* $g(A'') = o$. *Since* $F_o \subseteq A''$ *follows* $g(F_o) = o$. *Let* $y \in F$, *then* $y = y_1 + y_2$ *where* $y_1 \in F_o, y_2 \in P$, *therefore*

$$g(y) = g(y_1) + g(y_2) = g(y_2)$$

but $ge(y) = ge(y_1) + ge(y_2) = ge(y_2) = g(y)$. *Thus* $g(y) = ge(y)$ *for all* $y \in F$. *From this fact follows* $g = ge \in Se$. *Thus our proof is completed.*

Proposition 5. *Let* F *be a free* R*-module and* $I = \{x_\alpha\}$ *be a subset of* $S = End_R(F)$, *then the following conditions are equivalent:*

(1)- *There exists an idempotent* f *of* S *such that* $r(I) \subseteq fS$.

(2)- *There exists an idempotent* e *of* S *such that* $e \in l(I)$.

(3)- *The submodule* $A = \sum_{x_\alpha \in I} \operatorname{Im} x_\alpha$ *contained in a direct summand of* F.

Proof. $(1)\Leftrightarrow(2)$ *it follows from lemma 1.*

$(2)\Rightarrow(3)$. *Let* e *be an idempotent of* S *such that* $e \in l(I)$, *and let* $x_\alpha \in I$, *then* $ex_\alpha = o$, *so* $\operatorname{Im} x_\alpha \subseteq Ker e$ *for all* $x_\alpha \in I$. *Thus* $A = \sum_{x_\alpha \in I} \operatorname{Im} x_\alpha \subseteq Ker e$, *where Ker e is a direct summand of* F.

$(3)\Rightarrow(2)$. *Assume* $A = \sum_{x_\alpha \in I} \operatorname{Im} x_\alpha \subseteq F_0$ *where* F_o *a direct summand of* F. *Then* $F = F_o \oplus P$ *where* P *a submodule of* F. *Suppose* $f : F \to P$ *the projection of* F *on to* P. *Thus* $f(F_o) = 0$, $f(A) = o$ *by lemma 3(2)* $f \in N(A) = l(I)$.

Proposition 6. *Let F be a free R-module and $I = \{x_\alpha\}$ be a subset of $S = End_R(F)$, then the following conditions are equivalent:*

(1)- *There exists an idempotent f of S such that $f \in r(I)$.*

(2)- $B = \bigcap_{x_\alpha \in I} Kerx_\alpha$ *contains a direct summand of F.*

Proof. (1)$\Rightarrow$(2). *Let f be an idempotent of S such that $f \in r(I)$, by lemma 3(2) $f \in r(I) = Q(B)$ where $B = \bigcap_{x_\alpha \in I} Kerx_\alpha$. Thus* $\operatorname{Im} f \subseteq B$ *and Imf is a direct summand of F.*

(2)$\Rightarrow$(1). *Let F_o be a direct summand of F such that $F_o \subseteq B$, then $F = F_o \oplus P$ where P a submodule of F. Suppose f the projection of F onto F_o, then f is an idempotent of S and* $\operatorname{Im} f = F_o \subseteq B$. *Thus by lemma 3(2) $f \in Q(B) = r(I)$.*

Theorem 7. *Let F be a free R-module and $S = End_R(F)$, then the following conditions are equivalent:*

(1)- $(\sum_{x_\alpha \in I} \operatorname{Im} x_\alpha)''$ *contains a direct summand of F, for any subset $I = \{x_\alpha\}$ of S.*

(2)- *The submodule $\bigcap_{x_\alpha \in I} Kerx_\alpha$ contains a direct summand of F, for any subset $I = \{x_\alpha\}$ of S.*

(3)- *Every closed submodule of F contains a direct summand of F.*

(4)- *The ring S is a generalized right Baer ring.*

Proof. (1)$\Leftrightarrow$(2) *it follows from proposition 4.*

(2)$\Leftrightarrow$(4) *it is clear by using proposition 6.*

$(3)\Rightarrow(4)$. *Let* $B=\bigcap_{x_\alpha\in I} Kerx_\alpha$ *, then by lemma 3(3),* $B=K(N(B))$ *therefore* $B=B''$*, thus* B *is closed submodule of* F*, by our hypothesis* B *contains a direct summand of* F*. by proposition 6 there exists an idempotent* f *of* S *such that* $f\in r(I)$*. Thus, we have that* S *a generalized Baer ring.*

$(4)\Rightarrow(3)$. *Let* A *be a closed submodule of* F*, then* $A=A''$*, by lemma 3,* $A=A''=K(N(A))$*, therefore* $B=\bigcap_{x_\alpha\in I} Kerx_\alpha$ *for some subset* $I=\{x_\alpha\}$ *of* S*. Since* S *is a generalized Baer ring there exists an idempotent* e *of* S *such that* $e\in r(I)$*. Thus by proposition 6,* $B=\bigcap_{x_\alpha\in I} Kerx_\alpha$ *contains a direct summand of* F*.*

Theorem 8. *Let* F *be a free* R*-module and* $S=End_R(F)$*, then the following conditions are equivalent:*

(1)- *The submodule* $\sum_{x_\alpha\in I}\operatorname{Im} x_\alpha$ *contained in a direct summand of* F*, for any subset* $I=\{x_\alpha\}$ *of* S*.*

(2)- *Every submodule of* F *contained in a direct summand of* F*.*

(3)- *The ring* S *is a generalized Baer ring.*

Proof. $(1)\Leftrightarrow(3)$ *it follows from proposition 5.*

$(2)\Rightarrow(1)$ *it is clear.*

$(1)\Rightarrow(2)$ *it follows from the fact, that any submodule of* F *can be considered as sum of images of some subset of* S*.*

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