Extremal Topologies

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ABSTRACT

If X is a set, τ is not a discrete topology on X then τ is called an extremal topology if every topology which is strictly finer than τ is discrete.

The main purpose of this paper is to prove an existence theorem for extremal topologies and to prove a second theorem, which determines how an extremal topology on a finite set looks. By using these two theorems we prove a counting theorem which gives the number of extremal topologies on a set with n elements.

Key Words: Topology, Extremal topology.

التوبولوجيـات المتطرفـة

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الملخص

إذا كانت X مجموعة , τ توبولوجيا غير متقطعة على X فإن τ تسمى توبولوجيا متطرفة إذا كانـت كل توبولوجيا تحتوى τ احتواء فعلياً تكون متقطعة.

الهدف الرئيس لهذه الورقة هو إثبات مبرهنة وجود للتوبولوجيات المتطرفة ثم إثبات مبرهنة ثانية يمكن بواسطتها تحديد شكل التوبولوجيات المتطرفة على مجموعة منتهية وباستخدام هاتين المبـرهنتين يتم إثبات مبرهنة تعطي عدد التوبولوجيات المتطرفة على مجموعة عدد عناصرها n .

الكلمات المفتاحية: توبولوجيا، توبولوجيا متطرفة.

0- Preliminaries.

0-1 If $A\subset X$, τ is a topology on X then $\tau\langle A\rangle=\{U\cup(V\cap A):\ U, V\in\tau\}$ is a topology on X finer than τ and containing the set A, and it is called the simple extension of τ over A. see 3A.5 of [1]

0-2 Lemma:-

If X is a finite set with n elements and $x, y\in X$, $x\neq y$ then:

$$\left|\{A\subset X:\ \ x\notin A, y\in A\}\right| = 2^{n-2}\,,\quad \text{where } n\geq 2.$$

Proof:

Let $A\subset X$, $x\notin A$ then $A\subset X\setminus\{x\}$. Let φ: $\{A:\ x\notin A,\ y\in A\}\to\{B:\ B\subset X\setminus\{x,y\}\}$ defined by $\varphi(A)=A\setminus\{y\}$ then φ is 1-1 map, and if $B\subset X\setminus\{x,y\}$

Then $B\cup\{y\}\in\{A:\ x\notin A,\ y\in A\}$ and $\varphi(B\cup\{y\})=B$ so φ is 1-1 and onto,

And hence

$$\left|\{A\subset X:\ \ x\notin A, y\in A\}\right| = \left|\{B:\ \ B\subset X\setminus\{x,y\}\}\right| = \left|P(X\setminus\{x,y\})\right| = 2^{n-2}.$$

0-3 There are $\binom{n}{k}$ ways to choose k different elements from a set with n elements where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

1-Extremal Topology.

In [2] T. papzyan defined extremal topologies for a class of topological spaces defined on a semigroup, in this section we define extremal topologies on an arbitrary set and we will prove an existence theorem.

1-1 Definition:-

Let X be any set, τ is not a discrete topology on X then τ is said to be an extremal topology if every topology strictly finer than τ is discrete.

1-2 Theorem:-

If X is any set with more than one element, x, $y\in X$, $x\neq y$, and $\tau_{\{x,y\}}=P(X\backslash\{x\})\cup\{\{x\}\cup A : A\in P(X\backslash\{x\}),\ y\in A\}$ then $\tau_{\{x,y\}}$ is an extremal topology on X.

Proof:

Clearly $\emptyset\in\tau_{\{x,y\}}$ and if $A=X\backslash\{x\}$ then $y\in A\subset X\backslash\{x\}$, and so $X=\{x\}\cup A\in\tau_{\{x,y\}}$. Let U_1, $U_2\in\tau_{\{x,y\}}$. If U_1, $U_2\in P(X\backslash\{x\})$ then $U_1\cap U_2\in P(X\backslash\{x\})\subseteq\tau_{\{x,y\}}$.

If $U_1=\{x\}\cup A_1$, $U_2=\{x\}\cup A_2$ where A_1, $A_2\in P(X\backslash\{x\})$ and $y\in A_1\cap A_2$. Then $U_1\cap U_2=\{x\}\cup(A_1\cap A_2)\in\tau_{\{x,y\}}$. If $U_1\in P(X\backslash\{x\})$, $U_2=\{x\}\cup A$ and $y\in A\in P(X\backslash\{x\})$ Then $U_1\cap U_2=U_1\cap A\in P(X\backslash\{x\})$ so $U_1\cap U_2\in\tau_{\{x,y\}}$.Therefore $U_1\cap U_2\in\tau_{\{x,y\}}$ for all U_1, $U_2\in\tau_{\{x,y\}}$.

Let $U_i\in\tau_{\{x,y\}}$ for all $i\in I$. If $U_i\in P(X\backslash\{x\})$ for all $i\in I$. Then $\bigcup_{i\in I}U_i\in P(X\backslash\{x\})\subset\tau_{\{x,y\}}$.

If $U_i=\{x\}\cup A_i$ where $A_i\in P(X\backslash\{x\})$, $y\in A_i$ then

$$\bigcup_{i\in I} A_i \in P(X \setminus \{x\}), \quad y \in \bigcup_{i\in I} A_i \text{ and so } \bigcup_{i\in I} U_i \in \tau_{\{x,y\}}.$$

If there exists $j\in I$ such that $U_j=\{x\}\cup A$, $A\in P(X\setminus\{x\})$, $y\in A$. Then $\bigcup_{i\in I} U_i=\{x\}\cup B$ where $B=\bigcup_{i\in I}(U_i \setminus \{x\}) \in P(X \setminus \{x\})$, $y\in B$. So $\bigcup_{i\in I} U_i \in\tau_{\{x,y\}}$. And hence $\tau_{\{x,y\}}$ is a topology. To show $\tau_{\{x,y\}}$ is an extremal if $\tau_{\{x,y\}}\subset\tau$ and $\tau_{\{x,y\}}\neq\tau$. Then there exists $U\in\tau$ such that $U\notin\tau_{\{x,y\}}$, so $U\notin P(X\setminus\{x\})$ and $U\neq\{x\}\cup A$ for any $A\in P(X\setminus\{x\})$ with $y\in A$. Hence $U=\{x\}$ or $U=\{x\}\cup B$ where $B\in P(X\setminus\{x,y\})$. If $U=\{x\}$ since $P(X\setminus\{x\})\subseteq\tau$ then $\tau=P(X)$ that is τ is discrete. If $U=\{x\}\cup B$ where $B\in P(X\setminus\{x,y\})$ then since $\{x,y\}\in\tau$ so $U\cap\{x,y\}=\{x\}\in\tau$ and hence τ is discrete. Therefore $\tau_{\{x,y\}}$ is an extremal topology .

1-3 Remarks:-

i- Notice that if X is a set, $x, y\in X$, $x\neq y$ then $\tau_{\{x,y\}}\neq\tau_{\{y,x\}}$.

ii- If $B\subset X$ then $\tau_{\{x,B\}}=P(X\setminus\{x\})\cup\{\{x\}\cup A\colon B\subset A\subset X\setminus\{x\}\}$ is a topology on X, and it is an extremal if and only if B is a one- point set.

2- Extremal topology on a finite set.

In the following theorem we prove that every extremal topology on a finite set has to be in a certain form.

2-1 Theorem:-

Any extremal topology on a finite set with more than one element is in the form $\tau_{\{x,y\}}$ for some x, $y\in X$, $x\neq y$.

Proof:

Let τ be any extremal topology on X, where X is finite with more than one element . Then τ is not discrete, so there exists $x\in X$ such that $\{x\}\notin\tau$.

Now if there exists $y\in X\backslash\{x\}$ such that $\{y\}\notin\tau$. Then let $\tau\langle\{y\}\rangle$ be the simple extension of τ over $\{y\}$. Then $\tau\subset\tau\langle\{y\}\rangle$. Since τ is extremal so $\tau\langle\{y\}\rangle$ is discrete and so there are U, $V\in\tau$ such that $\{x\}=U\cup(V\cap\{y\})$. But then $U=\{x\}$ which is a contradiction. Hence $\{y\}\in\tau$ for all $y\in X\backslash\{x\}$ and so $P(X\backslash\{x\})\subset\tau$.

Now let $\wp=\{A\in P(X\backslash\{x\}): \{x\}\cup A\in\tau\}$ since $X\backslash\{x\}\in\wp$ so $\wp\neq\emptyset$. Let $A_0=\bigcap_{A\in\wp} A$ then $A_0\neq\emptyset$ and $A_0\in\wp$, and since $A_0\subset A$ for all $A\in\wp$, so $\tau\subset\tau_{\{x,A_0\}}$, and since $A_0\in\wp$ so $\tau_{\{x,A_0\}}\subset\tau$ and hence $\tau=\tau_{\{x,A_0\}}$. Since τ is extremal A_0 must be a single point. That is there exists $y\in X$ such that $\tau=\tau_{\{x,y\}}$.

2-2 Theorem:-

If X is a set with n elements then the number of extremal topologies defined on X is $n(n-1)$.

Proof:

By theorem (2-1) every extremal topology has the form $\tau_{\{x,y\}}$ for some x, $y\in X$ and since $\tau_{\{x,y\}}\neq\tau_{\{y,x\}}$ for all $x,y\in X$, $x\neq y$. So any two distinct elements determine two distinct topologies on X. Therefore the number of extremal topologies defined on a set with n elements equal

$$2\binom{n}{2} = 2\frac{n!}{2!(n-2)!} = n(n-1).$$

2-3 Theorem:-

If X is a set with n elements then any extremal topology has $3(2^{n-2})$ elements.

Proof:

By theorem 2-1 any extremal topology has the form $\tau_{\{x,y\}}$ for some x, $y\in X$ and since $\tau_{\{x,y\}}=P(X\backslash\{x\})\cup\{\{x\}\cup A: A\in P(X\backslash\{x\}), y\in A\}$ so

$|\tau_{\{x,y\}}| = |P(X\backslash\{x\})| + |\{\{x\}\cup A: A\in P(X\backslash\{x\}), y\in A\}|$. So by lemma 0-2

$|\tau_{\{x,y\}}| = 2^{n-1}+2^{n-2}=2^{n-2}(2+1)=3(2^{n-2})$.

2-4 Remark:-

As the reader noticed theorem 2-2 was proved only for finite sets and we don't have an example of an extremal topology on an infinite set which is not in the form $\tau_{\{x, y\}}$.

REFERENCES

[1] Willard, S. General topology (Addison-wesley, Reading, MA 1970).

[2] Papazyan, T. Extremal topologies on a semigroup, Topology and its applications 39 (1991) 229-243.