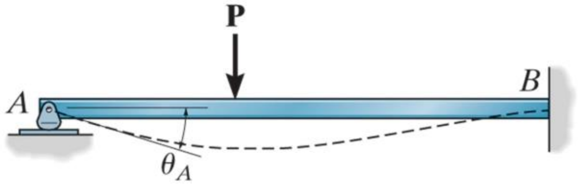
طرائق الانتقالات في التحليل الإنشائي
Displacement Methods of Analysis

طريقة الميل والسهم
Slope-Deflection Equations

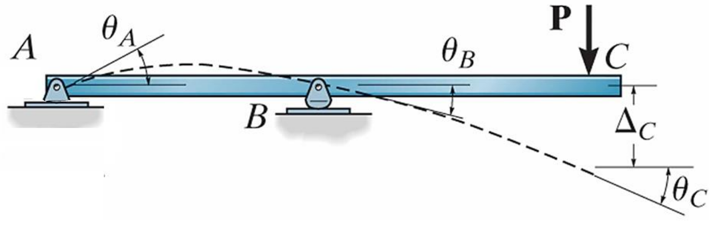
Degrees of Freedom درجات الحرية

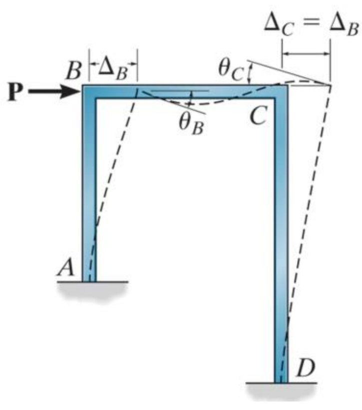
- عندما يتم تحميل المنشأ تخضع نقاط محددة عليه تُدعى العقد إلى انتقالات مجهولة تسمى درجات الحرية، يُشار إلى عدد غير المقيدة منها بدرجة عدم التقرير الحركي
- ولتحديد درجة عدم التقرير الحركي يتم تخيل المنشأ على أنه يتألف من مجموعة من العناصر متصلة بالعقد، ويحدد مواقع العقد عند اطراف العناصر وعند المساند وعند تغير عطالة العناصر
- تمتلك كل عقدة ست درجات حرية في المنشآت ثلاثية البعد وثلاث درجات حرية في المنشآت ثنائية البعد

- يسبب الحمل P في الجائز المبين دوران θ_A في العقدة A في حين العقدة B ممنوعة من الانتقال والدوران، وعلية فللجائز درجة حرية واحدة إذا أهملت التشوهات الطولية



- للجائز المبين في الشكل ثلاث عقد A, B, C ويمتك ثلاث درجات حرية دورانية $\theta_A, \theta_B, \theta_C$ وانتقال شاقولي Δ_C وعليه للجائز درجة عدم تقرير حركي تساوي أربعة



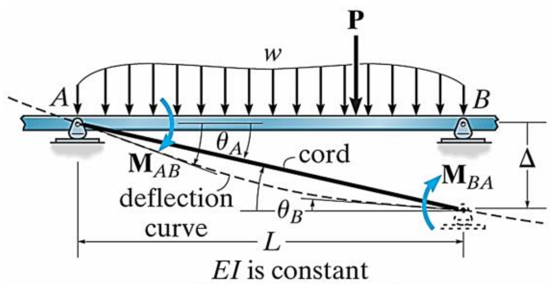


- يسبب الحمل المطبق على الإطار المبين في الشكل دورانين في كل من العقدتين وانتقال افقي لكل منهما، هذين الانتقالين متساويين إذا أهملت التشوهات المحورية وعليه للاطار درجة عدم تقرير حركي تساوي ثلاثة

- إذاً يُعد تحديد درجة عدم التقرير الحركي (عدد درجات الحرية غير المقيدة) الخطوة الضرورية الأولى لطرائق الانتقالات في التحليل الإنشائي، لأنه ما إن تُعرف درجات الحرية (الانتقالات) تصبح تشوهات عناصر المنشأ محددة وعليه يمكن حساب القوى الداخلية في هذه العناصر

Slope-Deflection Equations معادلات الميل والسهم

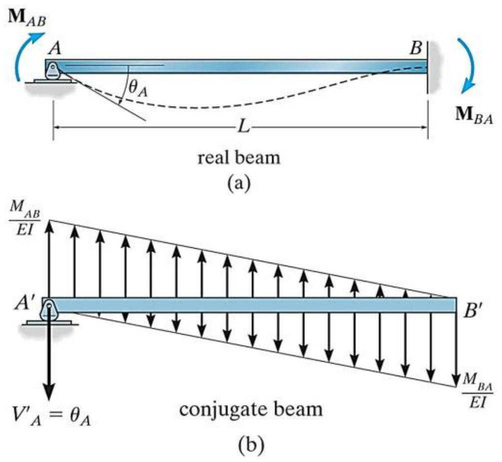
- لندرس المجاز AB من الجائز المبين في الشكل وذلك لإيجاد العلاقة بين العزمين الداخليين M_{AB} , M_{BA} ودرجات الحرية Δ , θ_A, , θ_B,



EI is constant
positive sign convention

اصطلاح الإشارة:
تُعد العزوم المطبقة على العناصر موجبة إذا كانت باتجاه عقارب الساعة وتعد سالبة إذا كانت عكس ذلك.

Angular Displacement θ_A زاوية الدوران في A



- لتحديد العزم M_{AB} اللازم لإحداث زاوية الدوران θ_A في الجائز المبين في الشكل a نستخدم طريقة الجائز المرافق وعليه يبين الشكل b الجائز المرافق للجائز في الشكل a
- بما أن زاوية الدوران θ_A مع عقارب الساعة فإن جهة القص عند A' ستكون للأسفل

بكتابة معادلة توازن عزوم الجائز المرافق عند A' و B' نجد:

$$\curvearrowright + \Sigma M_{A'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$$

$$\curvearrowright + \Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$$

الحل المشترك لهاتين المعادلتين يُعطي

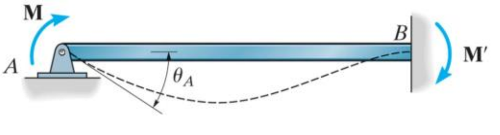
$$M_{AB} = \frac{4EI}{L}\theta_A$$

$$M_{BA} = \frac{2EI}{L}\theta_A$$

معامل صلابة عنصر :Member Stiffness Factor

يسبب تطبيق عزم M عند الطرف A العنصر المبين في الشكل دوراناً قدره θ_A

$$M = (4EI / L)\, \theta_A$$



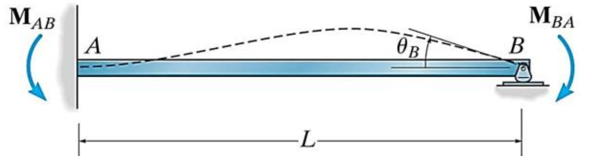
يدعى $4EI\ /\ L$ بمعامل الصلابة عند A ويعرف بأنه العزم M اللازم لإحداث واحدة الدوران عند A

$$K = \frac{4EI}{L}$$

Far End Fixed

Angular Displacement θ_B B زاوية الدوران في

- بشكل مشابه لما ورد في الحالة السابقة إذا خضعت النهاية B من الجائز المبين في الشكل لعزم M_{BA} فسينتج دوران θ_B

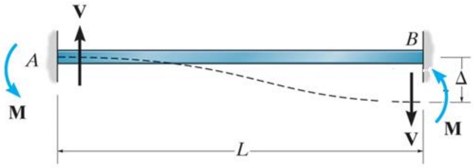


$$M_{BA} = \frac{4EI}{L}\theta_B \quad (11\text{–}3)$$

$$M_{AB} = \frac{2EI}{L}\theta_B \quad (11\text{–}4)$$

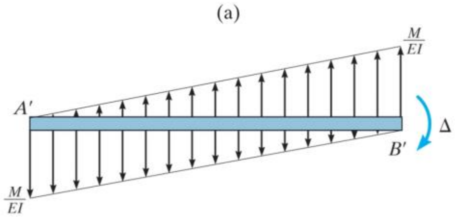
حالة انتقال نسبي Δ بين طرفي جائز موثوق

- إذا انتقل الطرف B من الجائز المبين في الشكل a بمقدار Δ بالنسبة للطرف A فسينشأ في طرفي الجائز عزم وقص كما في الشكل.
- يبين الشكل b الجائز المرافق
- بما أن طرف الجائز B انتقل بمقدار Δ فسيخضع طرف الجائز المرافق إلى عزم مقداره Δ



real beam

(a)



conjugate beam

(b)

- بكتابة معادلة توازن عزوم الجائز المرافق عند B' نجد:

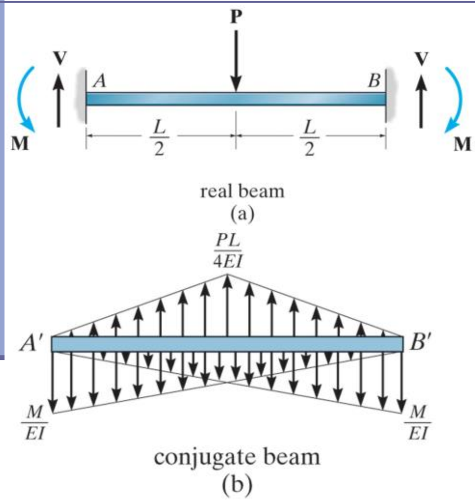
$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{2}{3}L\right)\right] - \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{1}{3}L\right)\right] - \Delta = 0$$

$$\boxed{M_{AB} = M_{BA} = M = \frac{-6EI}{L^2}\Delta}$$

عزوم الوثاقة (Fixed-End Moments (FEMs:

- عموماً تنتج الانتقالات الخطية والدورانية في العقد عن الأحمال في المجازات ولهذا لا بد من تحويل أحمال المجازات إلى عزوم مكافئة مطبقة في العقد وذلك لاستخدامها في العلاقات التي تم اشتقاقها في الفقرات السابقة

عزوم الوثاقة (Fixed-End Moments (FEMs: يُسمى العزم الناجم في طرف موثوق من عنصر إنشائي محمل بحمل ما بعزم الوثاقة



- لندرس الجائز الموثوق من طرفيه المبين في الشكل (a). وليكن الجائز المرافق له مبيناً في الشكل (b).

بكتابة معادلة التوازن على المحور الشاقولي للجائز المرافق ينتج:

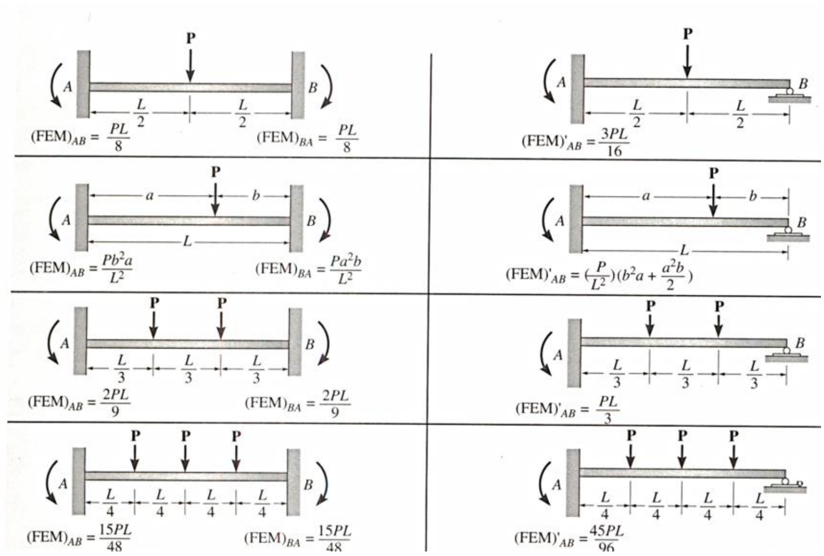
$$+\uparrow \Sigma F_y = 0; \qquad \left[\frac{1}{2}\left(\frac{PL}{4EI}\right)L\right] - 2\left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right] = 0$$

$$M = \frac{PL}{8}$$

يدعى هذا العزم بعزم الوثاقة (FEM) ويعطى إشارة سالبة عند الطرف A لأنه عكس عقارب الساعة ويعطى إشارة موجبة عند الطرف B لأنه مع عقارب الساعة.

يبين الجدول التالي عزوم الوثاقة لبعض حالات التحميل الشهيرة

$$M_{AB} = (\mathrm{FEM})_{AB} \qquad M_{BA} = (\mathrm{FEM})_{BA}$$



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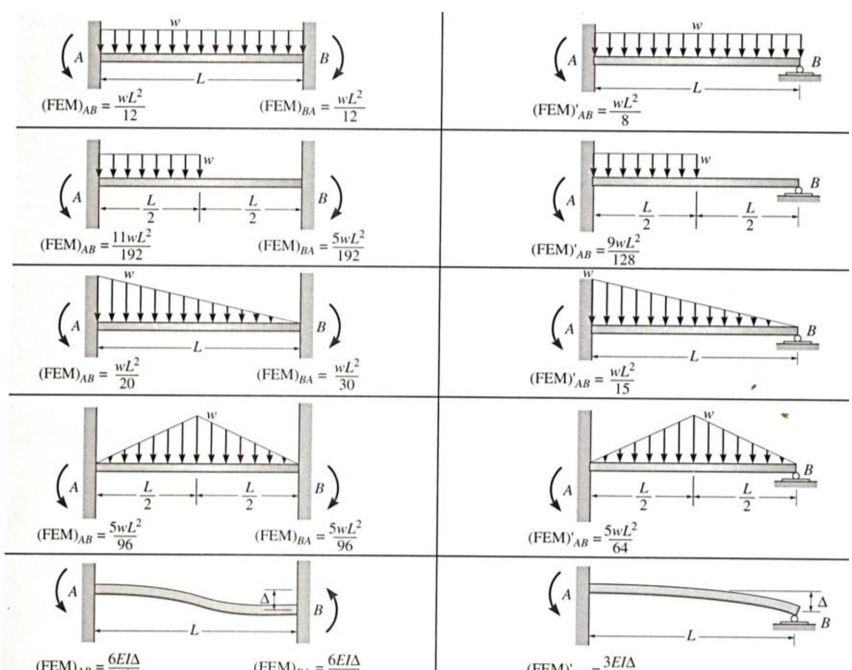

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Slope-Deflection Equation معادلة الميل والسهم

- إذا تم إضافة العزوم في كل طرف من الجائز الناتجة عن الانتقالات والأحمال ينتج:

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + (\mathrm{FEM})_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + (\mathrm{FEM})_{BA} \qquad (11\text{–}7)$$

Or

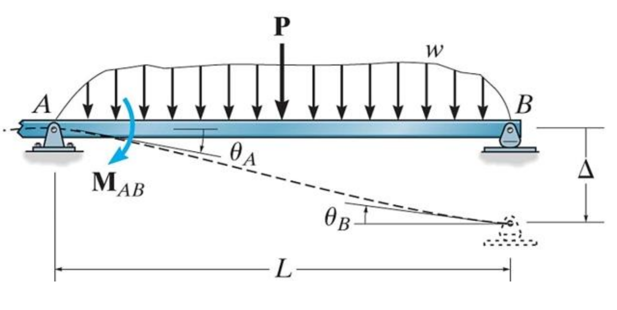
$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\mathrm{FEM})_N \qquad (11\text{–}8)$$

For Internal Span or End Span with Far End Fixed

(N) ترمز للطرف القريب من العنصر و *(F)* للطرف البعيد و $k = I/L$ صلابة العنصر و $\psi = \Delta/L$ زاوية دوران وتر العنصر

Pin-Supported End Span مجاز طرفي ذو استناد بسيط

- عندما يستند الطرف البعيد من الجائز على مسند ثابت أو متدحرج يكون العزم عند هذا المسند معدوماً وبالتالي تُعدل المعادلة العامة للميل والسهم لتطبق مرة واحدة



بتطبيق المعادلتين 7-11 و 8-11 عند كل من طرفي العنصر ينتج

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$
$$0 = 2Ek(2\theta_F + \theta_N - 3\psi) + 0 \qquad (11\text{–}9)$$

بطرح المعادلة الثانية من الأولى بعد ضربها بـ 2 ينتج:

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N \qquad (11\text{–}10)$$

Only for End Span with Far End Pinned or Roller Supported

EXAMPLE 11-1

Draw the shear and moment diagrams for the beam shown in Fig. 11–10*a*. *EI* is constant.

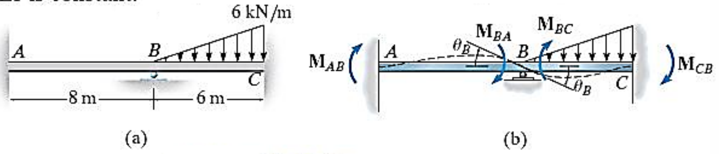


Fig. 11–10

Solution

Slope-Deflection Equations. Two spans must be considered in this problem. Since there is *no* span having the far end pinned or roller supported, Eq. 11–8 applies to the solution. Using the formulas for the FEMs tabulated for the triangular loading given on the inside back cover, we have

$$(\text{FEM})_{BC} = -\frac{wL^2}{30} = -\frac{6(6)^2}{30} = -7.2 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{20} = \frac{6(6)^2}{20} = 10.8 \text{ kN}\cdot\text{m}$$

EXAMPLE 11-1 (Continued)

Note that $(\text{FEM})_{BC}$ is negative since it acts counterclockwise *on the beam* at B. Also, $(\text{FEM})_{AB} = (\text{FEM})_{BA} = 0$ since there is no load on span AB.

In order to identify the unknowns, the elastic curve for the beam is shown in Fig. 11–10*b*. As indicated, there are four unknown internal moments. Only the slope at B, θ_B, is unknown. Since A and C are fixed supports, $\theta_A = \theta_C = 0$. Also, since the supports do not settle, nor are they displaced up or down, $\psi_{AB} = \psi_{BC} = 0$. For span AB, considering A to be the near end and B to be the far end, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{8}\right)[2(0) + \theta_B - 3(0)] + 0 = \frac{EI}{4}\theta_B \qquad (1)$$

Now, considering B to be the near end and A to be the far end, we have

$$M_{BA} = 2E\left(\frac{I}{8}\right)[2\theta_B + 0 - 3(0)] + 0 = \frac{EI}{2}\theta_B \qquad (2)$$

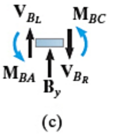
In a similar manner, for span BC we have

$$M_{BC} = 2E\left(\frac{I}{6}\right)[2\theta_B + 0 - 3(0)] - 7.2 = \frac{2EI}{3}\theta_B - 7.2 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_B - 3(0)] + 10.8 = \frac{EI}{3}\theta_B + 10.8 \qquad (4)$$

EXAMPLE 11-1 (Continued)

Equilibrium Equations. The above four equations contain five unknowns. The necessary fifth equation comes from the condition of moment equilibrium at support B. The free-body diagram of a segment of the beam at B is shown in Fig. 11–10*c*. Here $\mathbf{M}_{BA}$ and $\mathbf{M}_{BC}$ are assumed to act in the positive direction to be consistent with the slope-deflection equations.* The beam shears contribute negligible moment about B since the segment is of differential length. Thus,



(c)

$$\circlearrowleft + \Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (5)$$

To solve, substitute Eqs. (2) and (3) into Eq. (5), which yields

$$\theta_B = \frac{6.17}{EI}$$

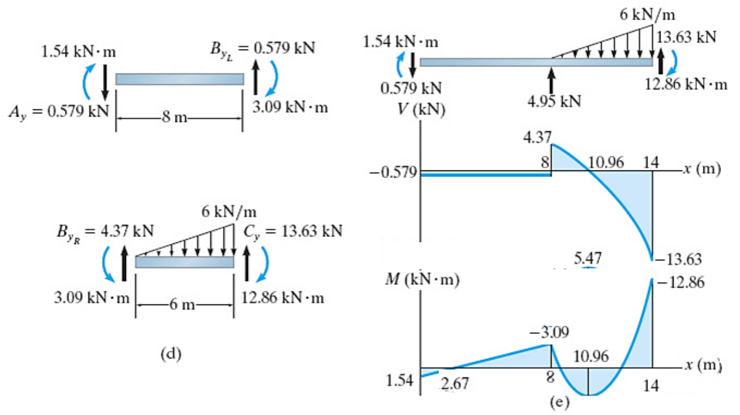
Resubstituting this value into Eqs. (1)–(4) yields

$$M_{AB} = 1.54\ \text{kN}\cdot\text{m}$$
$$M_{BA} = 3.09\ \text{kN}\cdot\text{m}$$
$$M_{BC} = -3.09\ \text{kN}\cdot\text{m}$$
$$M_{CB} = 12.86\ \text{kN}\cdot\text{m}$$

EXAMPLE 11-1 (Continued)

The negative value for M_{BC} indicates that this moment acts counterclockwise on the beam, not clockwise as shown in Fig. 11–10*b*.

Using these results, the shears at the end spans are determined from the equilibrium equations, Fig. 11–10*d*. The free-body diagram of the entire beam and the shear and moment diagrams are shown in Fig. 11–10*e*.



EXAMPLE 11-2

Draw the shear and moment diagrams for the beam shown in Fig. 11–11*a*. *EI* is constant.

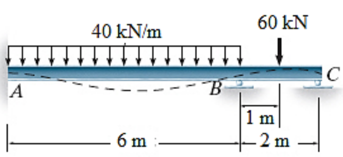


Fig. 11–11

(a)

Solution

Slope-Deflection Equations. Two spans must be considered in this problem. Equation 11–8 applies to span AB. We can use Eq. 11–10 for span BC since the *end C is on a roller*. Using the formulas for the FEMs tabulated on the inside back cover, we have

$$(\text{FEM})_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(40)(6)^2 = -120 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{BA} = \frac{wL^2}{12} = \frac{1}{12}(2)(40)(6)^2 = 120 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{BC} = -\frac{3PL}{16} = -\frac{3(60)(2)}{16} = -22.5 \text{ kN}\cdot\text{m}$$

Note that $(FEM)_{AB}$ and $(FEM)_{BC}$ are negative since they act counterclockwise on the beam at A and B, respectively. Also, since the supports do not settle, $\psi_{AB} = \psi_{BC} = 0$. Applying Eq. 11–8 for span AB and realizing that $\theta_A = 0$, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_B - 3(0)] - 120$$

$$M_{AB} = 0.3333EI\theta_B - 120 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{6}\right)[2\theta_B + 0 - 3(0)] + 120$$

$$M_{BA} = 0.667EI\theta_B + 120 \qquad (2)$$

Applying Eq. 11–10 with B as the near end and C as the far end, we have

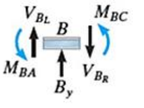
$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{BC} = 3E\left(\frac{I}{8}\right)(\theta_B - 0) - 22.5$$

$$M_{BC} = 1.5EI\theta_B - 22.5 \qquad (3)$$

EXAMPLE 11-2 *(Continued)*

Equilibrium Equations. The above three equations contain four unknowns. The necessary fourth equation comes from the conditions of equilibrium at the support B. The free-body diagram is shown in Fig. 11–11b. We have



(b)

$\curvearrowleft + \Sigma M_B = 0;$ $\qquad M_{BA} + M_{BC} = 0 \qquad (4)$

To solve, substitute Eqs. (2) and (3) into Eq. (4), which yields

$$\theta_B = -\frac{45}{EI}$$

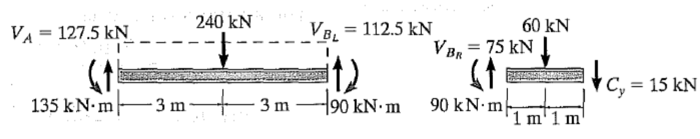
Since θ_B is negative (counterclockwise) the elastic curve for the beam has been correctly drawn in Fig. 11–11a. Substituting θ_B into Eqs. (1)–(3), we get

$$M_{AB} = -135 \text{ kN} \cdot \text{m}$$

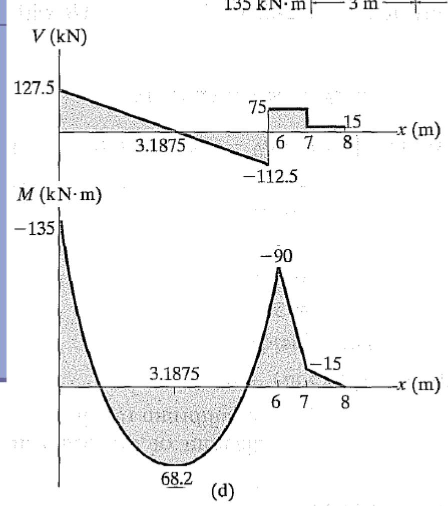
$$M_{BA} = 90 \text{ kN} \cdot \text{m}$$

$$M_{BC} = -90 \text{ kN} \cdot \text{m}$$

Using these data for the moments, the shear reactions at the ends of the beam spans have been determined in Fig. 11–11c. The shear and moment diagrams are plotted in Fig. 11–11d.



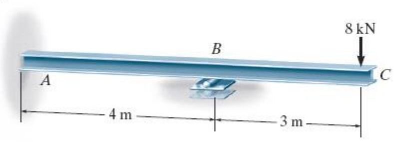
(c)



(d)

EXAMPLE 11-3

Determine the moment at A and B for the beam shown in Fig. 11–12a. The support at B is displaced (settles) 80 mm. Take $E = 200$ GPa, $I = 5(10^6)$ mm^4.

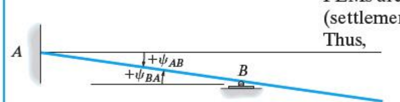


(a)

Fig. 11–12

Solution

Slope-Deflection Equations. Only one span (AB) must be considered in this problem since the moment $\mathbf{M}_{BC}$ due to the overhang can be calculated from statics. Since there is no loading on span AB, the FEMs are zero. As shown in Fig. 11–12b, the downward displacement (settlement) of B causes the cord for span AB to rotate clockwise. Thus,



(b)

$$\psi_{AB} = \psi_{BA} = \frac{0.08 \text{ m}}{4} = 0.02 \text{ rad}$$

The stiffness for AB is

$$k = \frac{I}{L} = \frac{5(10^6) \text{ mm}^4 (10^{-12}) \text{ m}^4/\text{mm}^4}{4 \text{ m}} = 1.25(10^{-6}) \text{ m}^3$$

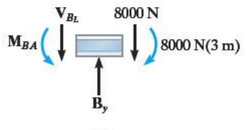
EXAMPLE 11-3 *(Continued)*

Applying the slope-deflection equation, Eq. 11–8, to span AB, with $\theta_A = 0$, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2(200(10^9)\ \text{N/m}^2)[1.25(10^{-6})\ \text{m}^3][2(0) + \theta_B - 3(0.02)] + 0 \quad (1)$$

$$M_{BA} = 2(200(10^9)\ \text{N/m}^2)[1.25(10^{-6})\ \text{m}^3][2\theta_B + 0 - 3(0.02)] + 0 \quad (2)$$



(c)

Equilibrium Equations. The free-body diagram of the beam at support B is shown in Fig. 11–12c. Moment equilibrium requires

$$\circlearrowleft + \Sigma M_B = 0; \qquad M_{BA} - 8000\ \text{N}(3\ \text{m}) = 0$$

Substituting Eq. (2) into this equation yields

$$1(10^6)\theta_B - 30(10^3) = 24(10^3)$$

$$\theta_B = 0.054\ \text{rad}$$

Thus, from Eqs. (1) and (2),

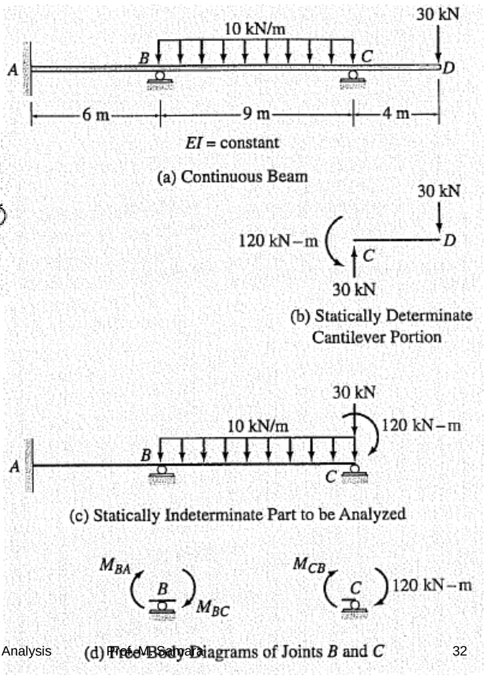
$$M_{AB} = -3.00\ \text{kN}\cdot\text{m}$$

$$M_{BA} = 24.0\ \text{kN}\cdot\text{m}$$

$$\text{FEM}_{AB} = \text{FEM}_{BA} = 0$$

$$\text{FEM}_{BC} = \frac{10(9)^2}{12} = 67.5\ \text{kN–m} ↺$$

$$\text{FEM}_{CB} = 67.5\ \text{kN–m} ↻$$



$$M_{AB} = \frac{2EI}{6}(\theta_B) = 0.333EI\theta_B$$

$$M_{BA} = \frac{2EI}{6}(2\theta_B) = 0.667EI\theta_B$$

$$M_{BC} = \frac{2EI}{9}(2\theta_B + \theta_C) - 67.5 = 0.444EI\theta_B + 0.222EI\theta_C - 67.5$$

$$M_{CB} = \frac{2EI}{9}(2\theta_C + \theta_B) + 67.5 = 0.222EI\theta_B + 0.444EI\theta_C + 67.5$$

$M_{BA} + M_{BC} = 0$	$1.111EI\theta_B + 0.222EI\theta_C = +67.5$	$EI\theta_B = +41.25$ kN–m^2
$M_{CB} + 120 = 0$	$0.222EI\theta_B + 0.444EI\theta_C = +52.5$	$EI\theta_C = +97.62$ kN–m^2

$M_{AB} = 0.333(+41.25) = +13.7$ kN–m or 13.7 kN–m ↻

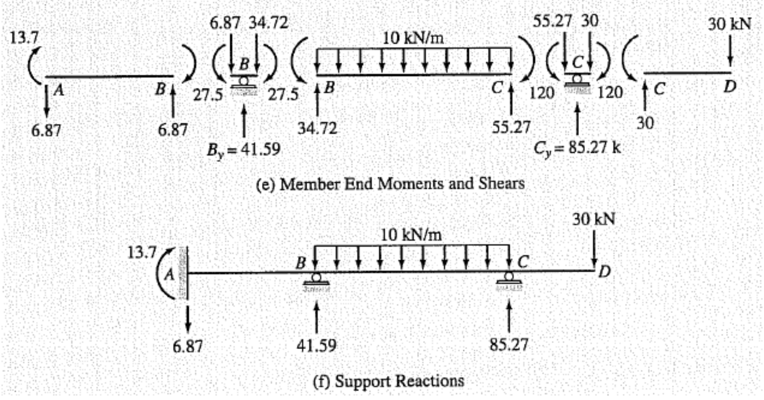
$M_{BA} = 0.667(+41.25) = +27.5$ kN–m or 27.5 kN–m ↻

$M_{BC} = 0.444(+41.25) + 0.222(+97.62) - 67.5$

$= 27.5$ kN–m ↺

$M_{CB} = 0.222(+41.25) + 0.444(+97.62) + 67.5$

$= +120$ kN–m or 120 kN–m ↻



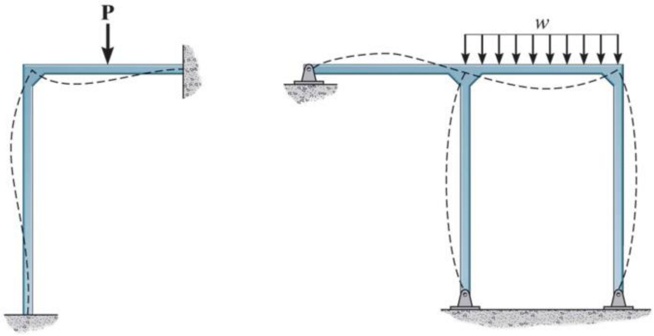
(e) Member End Moments and Shears

(f) Support Reactions

تحليل الإطارات بدون انزياح جانبي

Analysis of frames: No Sidesway

- يُقال عن اطار أنه لا يحتوي على انزياح جانبي إذا وجد قيد يمنعه من ذلك



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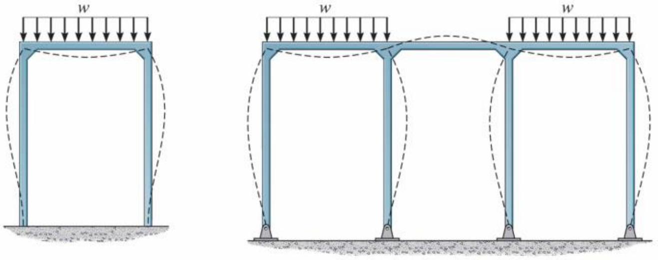
- لا يحدث انزياح جانبي في الاطار غير المقيد شريطة أن يكون متناظراً هندسياً وكذلك في التحميل



في كلا الحالتين السابقتين يتم تعويض الحد ψ بالصفر في معادلات الميل والسهم

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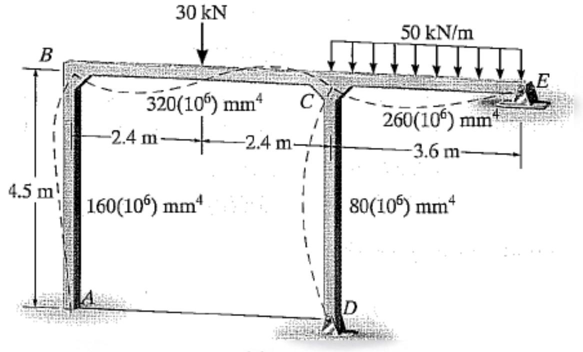
EXAMPLE 11-6

Determine the internal moments at each joint of the frame shown in Fig. 11–17*a*. The moment of inertia for each member is given in the figure. Take $E = 200$ GPa



(a)

Fig. 11–17

EXAMPLE 11-6 (Continued)

Solution

Slope-Deflection Equations. Four spans must be considered in this problem. Equation 11–8 applies to spans AB and BC, and Eq. 11–10 will be applied to CD and CE, because the ends at D and E are pinned.

Computing the member stiffnesses, we have

$$k_{AB} = \frac{160(10^6)(10^{-12})}{4.5} = 35.56(10^{-6})\ \text{m}^3 \qquad k_{CD} = \frac{80(10^6)(10^{-12})}{4.5} = 17.78(10^{-6})\ \text{m}^3$$

$$k_{BC} = \frac{320(10^6)(10^{-12})}{4.8} = 66.67(10^{-6})\ \text{m}^3 \qquad k_{CE} = \frac{260(10^6)(10^{-12})}{3.6} = 72.23(10^{-6})\ \text{m}^3$$

The FEMs due to the loadings are

$$(\text{FEM})_{BC} = -\frac{PL}{8} = -\frac{30(4.8)}{8} = -18\ \text{kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{PL}{8} = \frac{30(4.8)}{8} = 18\ \text{kN}\cdot\text{m}$$

$$(\text{FEM})_{CE} = -\frac{wL^2}{8} = -\frac{50(3.6)^2}{8} = -81\ \text{kN}\cdot\text{m}$$

Applying Eqs. 11–8 and 11–10 to the frame and noting that $\theta_A = 0$, $\psi_{AB} = \psi_{BC} = \psi_{CD} = \psi_{CE} = 0$ since no sidesway occurs, we have

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2[200(10^6)](35.56)(10^{-6})[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 14\ 222.2\theta_B \qquad (1)$$

EXAMPLE 11-6 (Continued)

$$M_{BA} = 2[200(10^6)](35.56)(10^{-6})[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 28\,444.4\theta_B \quad (2)$$

$$M_{BC} = 2[200(10^6)](66.67)(10^{-6})[2\theta_B + \theta_C - 3(0)] - 18$$

$$M_{BC} = 53\,333.3\theta_B + 26\,666.67\theta_C - 18 \quad (3)$$

$$M_{CB} = 2[200(10^6)](66.67)(10^{-6})[2\theta_C + \theta_B - 3(0)] + 18$$

$$M_{CB} = 26\,666.67\theta_B + 53\,333.3\theta_C + 18 \quad (4)$$

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{CD} = 3[200(10^6)](17.78)(10^{-6})[\theta_C - 0] + 0 \quad (5)$$

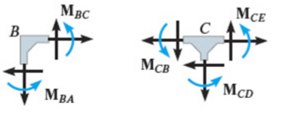
$$M_{CD} = 10\,666.7\theta_C$$

$$M_{CE} = 3[200(10^6)](72.22)(10^{-6})[\theta_C - 0] - 81$$

$$M_{CE} = 43\,333.3\theta_C - 81 \quad (6)$$

EXAMPLE 11-6 (Continued)

Equations of Equilibrium. These six equations contain eight unknowns. Two moment equilibrium equations can be written for joints B and C, Fig. 11–17b. We have



(b)

$$M_{BA} + M_{BC} = 0 \quad (7)$$

$$M_{CB} + M_{CD} + M_{CE} = 0 \quad (8)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), and Eqs. (4)–(6) into Eq. (8). This gives

$$81\,777.7\theta_B + 26\,666.7\theta_C = 18$$

$$26\,666.7\theta_B + 107\,333.3\theta_C = 63$$

Solving these equations simultaneously yields

$$\theta_B = 3.124(10^{-5}) \text{ rad} \qquad \theta_C = 5.792(10^{-4}) \text{ rad}$$

These values, being clockwise, tend to distort the frame as shown in Fig. 11–17a. Substituting these values into Eqs. (1)–(6) and solving, we get

$M_{AB} = 0.444$ kN · m *Ans.*

$M_{BA} = 0.888$ kN · m *Ans.*

$M_{BC} = -0.888$ kN · m *Ans.*

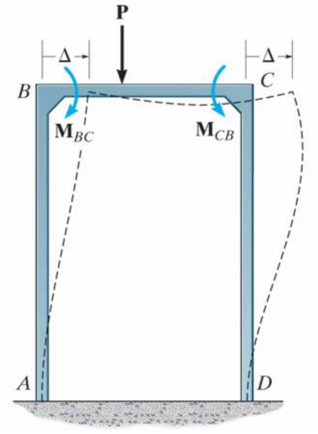
$M_{CB} = 49.7$ kN · m *Ans.*

$M_{CD} = 6.18$ kN · m *Ans.*

$M_{CE} = 55.9$ kN · m *Ans.*

تحليل الإطارات ذات الانزياح الجانبي

Analysis of frames: With Sidesway



- في الاطار المبين تسبب القوة P عزمين غير متساويين M_{BC} و M_{CB} في العقدتين B و C على الترتيب. يسعى العزم M_{BC} إلى تحريك العقدة B إلى اليمين في حين يسعى العزم M_{CB} إلى تحريك العقدة C إلى اليسار، وبما أن $M_{BC} > M_{CB}$ فستكون محصلة الانزياح الجانبي لكلا العقدتين إلى اليمين بمقدار Δ

- عند استخدام معادلة الميل والسهم على أعمدة الإطارات التي تشتمل على انزياح جانبي يجب ادخال دوران العمود ψ $(\psi = \Delta / L)$ كأحد مجاهيل المعادلة، ونتيجة لذلك يجب أن يشتمل الحل على معادلة توازن اضافية

EXAMPLE 11–8

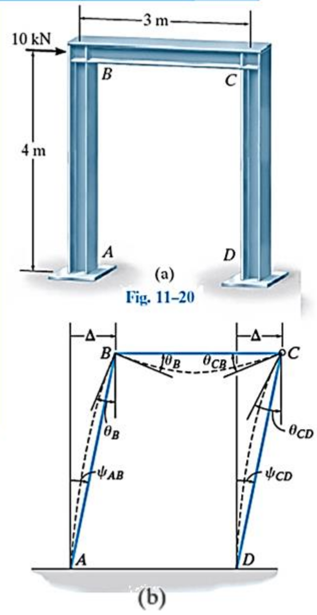


Fig. 11–20

Determine the moments at each joint of the frame shown in Fig. 11–20*a*. The supports at *A* and *D* are fixed and joint *C* is assumed pin connected. *EI* is constant for each member.

Solution

Slope-Deflection Equations. We will apply Eq. 11–8 to member *AB* since it is fixed connected at both ends. Equation 11–10 can be applied from *B* to *C* and from *D* to *C* since the pin at *C* supports zero moment. As shown by the deflection diagram, Fig. 11–20*b*, there is an unknown linear displacement Δ of the frame and unknown angular displacement θ_B at joint *B*.* Due to Δ, the cord members *AB* and *CD* rotate clockwise, $\psi = \psi_{AB} = \psi_{DC} = \Delta/4$. Realizing that $\theta_A = \theta_D = 0$ and that there are no FEMs for the members, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{4}\right)[2(0) + \theta_B - 3\psi] + 0 \qquad (1)$$

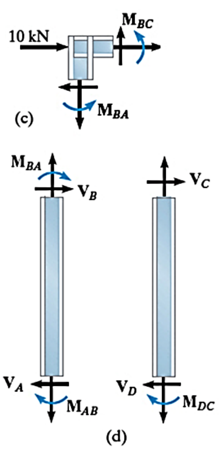
$$M_{BA} = 2E\left(\frac{I}{4}\right)(2\theta_B + 0 - 3\psi) + 0 \qquad (2)$$

EXAMPLE 11–8 *(Continued)*

$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{BC} = 3E\left(\frac{I}{3}\right)(\theta_B - 0) + 0 \qquad (3)$$

$$M_{DC} = 3E\left(\frac{I}{4}\right)(0 - \psi) + 0 \qquad (4)$$



Equilibrium Equations. Moment equilibrium of joint *B*, Fig. 11–20*c*, requires

$$M_{BA} + M_{BC} = 0 \qquad (5)$$

If forces are summed for the *entire frame* in the horizontal direction, we have

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 10 - V_A - V_D = 0 \qquad (6)$$

As shown on the free-body diagram of each column, Fig. 11–20*d*, we have

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{4}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC}}{4}$$

*The angular displacements θ_{CB} and θ_{CD} at joint *C* (pin) are not included in the analysis since Eq. 11–10 is to be used.

EXAMPLE 11-8 (Continued)

Thus, from Eq. (6),

$$10 + \frac{M_{AB} + M_{BA}}{4} + \frac{M_{DC}}{4} = 0 \qquad (7)$$

Substituting the slope-deflection equations into Eqs. (5) and (7) and simplifying yields

$$\theta_B = \frac{3}{4}\psi$$

$$10 + \frac{EI}{4}\left(\frac{3}{2}\theta_B - \frac{15}{4}\psi\right) = 0$$

Thus,

$$\theta_B = \frac{240}{21EI} \qquad \psi = \frac{320}{21EI}$$

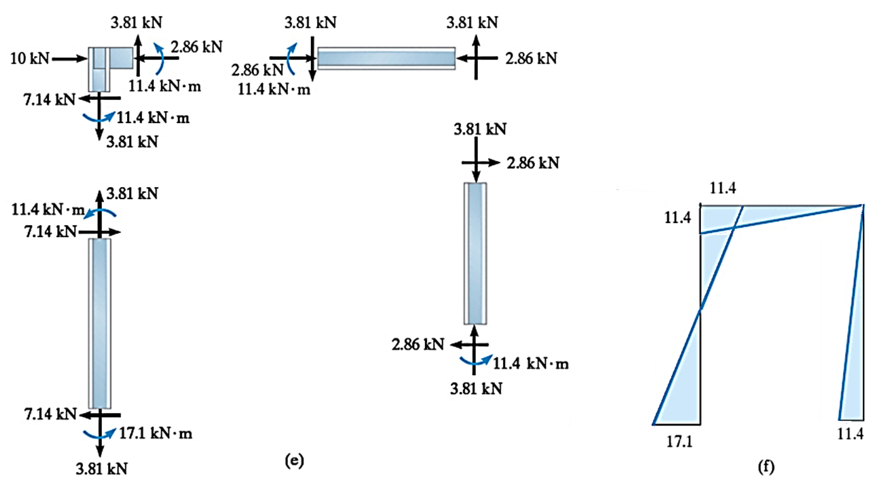
Substituting these values into Eqs. (1)–(4), we have

$M_{AB} = -17.1\ \text{kN}\cdot\text{m}$, $M_{BA} = -11.4\ \text{kN}\cdot\text{m}$ *Ans.*

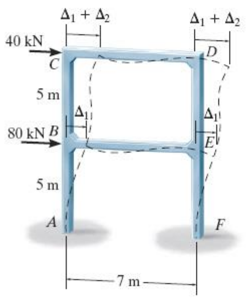
$M_{BC} = 11.4\ \text{kN}\cdot\text{m}$, $M_{DC} = -11.4\ \text{kN}\cdot\text{m}$ *Ans.*

EXAMPLE 11-8 (Continued)

Using these results, the end reactions on each member can be determined from the equations of equilibrium, Fig. 11–20*e*. The moment diagram for the frame is shown in Fig. 11–20*f*.



EXAMPLE 11-9



(a)

Fig. 11–21

Explain how the moments in each joint of the two-story frame shown in Fig. 11–21*a* are determined. *EI* is constant.

Solution

Slope-Deflection Equation. Since the supports at *A* and *F* are fixed, Eq. 11–8 applies for all six spans of the frame. No FEMs have to be calculated, since the applied loading acts at the joints. Here the loading displaces joints *B* and *E* an amount Δ_1, and *C* and *D* an amount $\Delta_1 + \Delta_2$. As a result, members *AB* and *FE* undergo rotations of $\psi_1 = \Delta_1/5$ and *BC* and *ED* undergo rotations of $\psi_2 = \Delta_2/5$.

Applying Eq. 11–8 to the frame yields

$$M_{AB} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_B - 3\psi_1] + 0 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{5}\right)[2\theta_B + 0 - 3\psi_1] + 0 \qquad (2)$$

$$M_{BC} = 2E\left(\frac{I}{5}\right)[2\theta_B + \theta_C - 3\psi_2] + 0 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{5}\right)[2\theta_C + \theta_B - 3\psi_2] + 0 \qquad (4)$$

EXAMPLE 11-9 *(Continued)*

$$M_{CD} = 2E\left(\frac{I}{7}\right)[2\theta_C + \theta_D - 3(0)] + 0 \qquad (5)$$

$$M_{DC} = 2E\left(\frac{I}{7}\right)[2\theta_D + \theta_C - 3(0)] + 0 \qquad (6)$$

$$M_{BE} = 2E\left(\frac{I}{7}\right)[2\theta_B + \theta_E - 3(0)] + 0 \qquad (7)$$

$$M_{EB} = 2E\left(\frac{I}{7}\right)[2\theta_E + \theta_B - 3(0)] + 0 \qquad (8)$$

$$M_{ED} = 2E\left(\frac{I}{5}\right)[2\theta_E + \theta_D - 3\psi_2] + 0 \qquad (9)$$

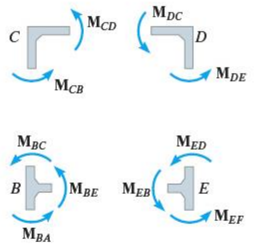
$$M_{DE} = 2E\left(\frac{I}{5}\right)[2\theta_D + \theta_E - 3\psi_2] + 0 \qquad (10)$$

$$M_{FE} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_E - 3\psi_1] + 0 \qquad (11)$$

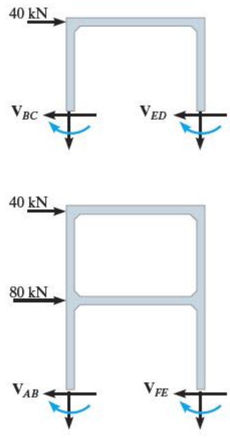
$$M_{EF} = 2E\left(\frac{I}{5}\right)[2\theta_E + 0 - 3\psi_1] + 0 \qquad (12)$$

These 12 equations contain 18 unknowns.

EXAMPLE 11-9 *(Continued)*



(b)



(c)

Equilibrium Equations. Moment equilibrium of joints $B, C, D,$ and E, Fig. 11–21b, requires

$$M_{BA} + M_{BE} + M_{BC} = 0 \quad (13)$$
$$M_{CB} + M_{CD} = 0 \quad (14)$$
$$M_{DC} + M_{DE} = 0 \quad (15)$$
$$M_{EF} + M_{EB} + M_{ED} = 0 \quad (16)$$

EXAMPLE 11-9 *(Continued)*

As in the preceding examples, the shear at the base of all the columns for any story must balance the applied horizontal loads, Fig. 11–21c. This yields

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 40 - V_{BC} - V_{ED} = 0$$

$$40 + \frac{M_{BC} + M_{CB}}{5} + \frac{M_{ED} + M_{DE}}{5} = 0 \quad (17)$$

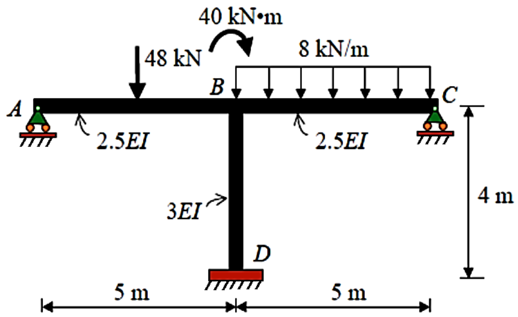
$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 40 + 80 - V_{AB} - V_{FE} = 0$$

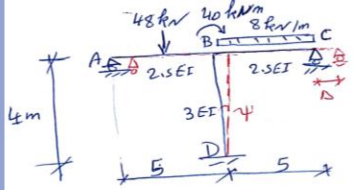
$$120 + \frac{M_{AB} + M_{BA}}{5} + \frac{M_{EF} + M_{FE}}{5} = 0 \quad (18)$$

Solution requires substituting Eqs. (1)–(12) into Eqs. (13)–(18), which yields six equations having six unknowns, ψ_1, ψ_2, θ_B, θ_C, θ_D, and θ_E. These equations can then be solved simultaneously. The results are resubstituted into Eqs. (1)–(12), which yields the moments at the joints.

Example

For the frame shown use the **slope-deflection** method to determine the end moments of the members and draw the bending moment diagram





$$FEM_{BA} = \frac{3PL}{16} = \frac{3 \times 48 \times 5}{16} = 45 \text{ kN m}$$

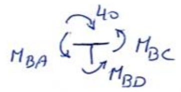
$$FEM_{BC} = -\frac{qL^2}{8} = -\frac{8 \times 25}{8} = -25 \text{ kN m}$$

$$M_{BA} = \frac{3 \times 2.5EI}{5}(\theta_B - 0) + 45 \Rightarrow M_{BA} = 1.5EI\theta_B + 45 \quad (1)$$

$$M_{BC} = \frac{3 \times 2.5EI}{5}(\theta_B - 0) - 25 \Rightarrow M_{BC} = 1.5EI\theta_B - 25 \quad (2)$$

$$M_{BD} = \frac{2 \times 3EI}{4}(2\theta_B + 0 - 3\psi) + 0 \Rightarrow M_{BD} = 3EI\theta_B - 4.5EI\psi \quad (3)$$

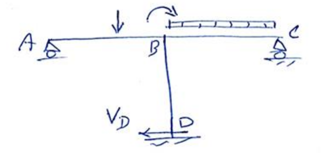
$$M_{DB} = \frac{2 \times 3EI}{4}(0 + \theta_B - 3\psi) + 0 \Rightarrow M_{DB} = 1.5EI\theta_B - 4.5EI\psi \quad (4)$$

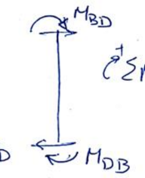


$$\Rightarrow M_{BA} + M_{BC} + M_{BD} = 40 \quad (5)$$

(6)

Substituting ①, ② and ③ into ⑤ ⇒ $6EI\theta_B - 4.5EI\psi - 20 = 0$ ⑥





$\sum M = 0 \Rightarrow V_D \times 4 + M_{BD} + M_{DB} = 0 \Rightarrow$

$\sum F_x = 0 \Rightarrow V_D = 0$

$M_{BD} + M_{DB} = 0$ ⑦

substituting ③ and ④ into ⑦ ⇒ $4.5EI\theta_B - 9EI\psi = 0$ ⑧

∴ solving ⑥ and ⑧ ⇒ $EI\theta_B = 5.334$, $EI\psi = 2.667$

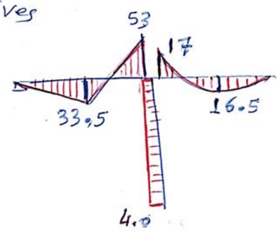
substituting this into ① → ④ gives

$M_{BA} = 53.0$ kN/m

$M_{BC} = -17.0$ kN/m

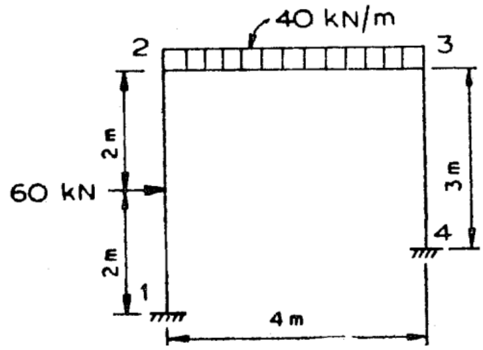
$M_{BD} = 4.0$

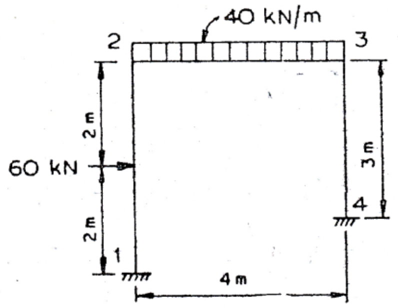
$M_{DB} = -4.0$

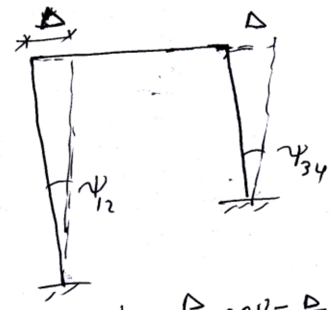


Example

Using the **slop-deflection** method determine the end moments of the members of the frame shown below and draw the bending moment diagram of the entire frame. *EI is* the same throughout.







$$\psi_{12} = \frac{\Delta}{4}, \quad \psi_{43} = \frac{\Delta}{3}$$

$$\psi_{12} = 0.75\,\psi_{43}$$

$$FEM_{12} = -FEM_{21} = -\frac{60 \times 4}{8} = -30 \text{ kN m}$$

$$FEM_{23} = -FEM_{32} = \frac{-40 \times 16}{12} = -53.33 \text{ kN m}$$

$$M_N = \frac{2EI}{\ell}(2\theta_N + \theta_F - 3\psi) + FEM_N$$

$$M_{12} = \frac{2EI}{4}(0 + \theta_2 - 3\psi_{12}) - 30 \Rightarrow M_{12} = 0.5EI\theta_2 - 1.125EI\psi_{43} - 30 \quad (1)$$

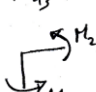
$$M_{21} = \frac{2EI}{4}(2\theta_2 - 3\psi_{12}) + 30 \Rightarrow M_{21} = EI\theta_2 - 1.125\,EI\psi_{43} + 30 \quad (2)$$

$$M_{23} = \frac{2EI}{4}(2\theta_2 + \theta_3) - 53.33 \Rightarrow M_{23} = EI\theta_2 + 0.5EI\theta_3 - 53.33 \quad (3)$$

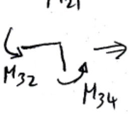
$$M_{32} = \frac{2EI}{4}(2\theta_3 + \theta_2) + 53.33 \Rightarrow M_{32} = 0.5EI\theta_2 + EI\theta_3 + 53.33 \quad (4)$$

$$M_{34} = \frac{2EI}{3}(2\theta_3 + 0 - 3\psi_{34}) \Rightarrow M_{34} = \frac{4EI}{3}\theta_3 - 2EI\psi_{34} \quad (5)$$

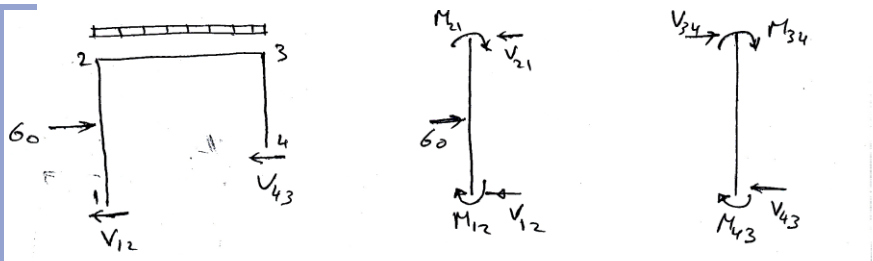
$$M_{43} = \frac{2EI}{3}(0 + \theta_3 - 3\psi_{34}) \Rightarrow M_{43} = \frac{2EI}{3}\theta_3 - 2EI\psi_{34} \quad (6)$$



$$\Rightarrow M_{21} + M_{23} = 0 \quad (7) \Rightarrow 2EI\theta_2 + 0.5EI\theta_3 - 1.125EI\psi_{34} - 23.33 = 0 \quad (8)$$



$$\Rightarrow M_{32} + M_{34} = 0 \quad (9) \Rightarrow 1.5EI\theta_2 + 7EI\theta_3 - 6EI\psi_{34} + 160 = 0 \quad (10)$$



$\Sigma F_x = 0 \Rightarrow V_{12} + V_{43} = 60$ (11)

$\overset{+}{\curvearrowleft} \Sigma M_2 = 0 \Rightarrow V_{12} = \frac{120 - M_{12} - M_{21}}{4}$ (12)

$\overset{+}{\curvearrowleft} \Sigma M_3 = 0 \Rightarrow V_{43} = \frac{-M_{34} - M_{43}}{3}$ (13)

Substituting eqn's 12 and 13 int eqn 11 $\Rightarrow$

$3(M_{12} + M_{21}) + 4(M_{34} + M_{43}) = -360$ (14)

Substituting (1), (2), (5) and (6) into (14) $\Rightarrow$

$4.5\,EI\theta_2 + 8\,EI\theta_3 - 22.75\,\psi_{34} = -360$ (15)

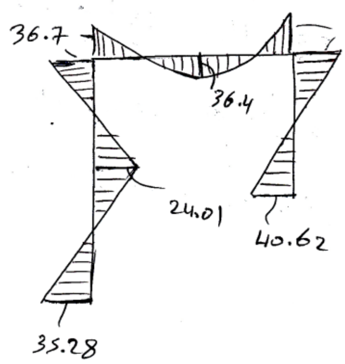
Solving eqn's (8), (10) and (15) $\Rightarrow$ $EI\theta_2 = 24.06$, $EI\theta_3 = -14.86$

$EI\psi_{34} = 15.34$

Substituting these into eqn's (1) $\rightarrow$ (6) $\Rightarrow$

$M_{12} = -35.28$ kNm , $M_{21} = 36.7$ kNm , $M_{23} = -36.7$ kNm

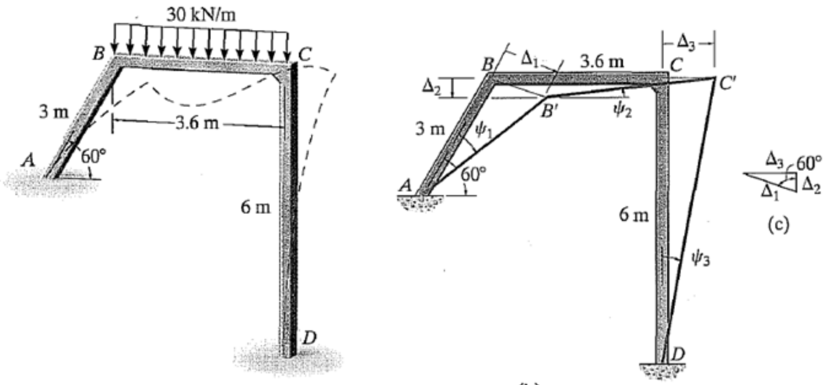
$M_{32} = 50.5$ kNm ; $M_{34} = -50.5$ kNm , $M_{43} = -40.62$



Bending Moment diagram

EXAMPLE 11-10

Determine the moments at each joint of the frame shown in Fig. 11–22*a*. *EI* is constant for each member.



(b)

Solution

Slope-Deflection Equations. Equation 11–8 applies to each of the three spans. The FEMs are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{30(3.6)^2}{12} = 32.4 \text{ kN} \cdot \text{m}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{30(3.6)^2}{12} = 32.4 \text{ kN} \cdot \text{m}$$

The sloping member *AB* causes the frame to sidesway to the right as shown in Fig. 11–22*a*. As a result, joints *B* and *C* are subjected to both rotational *and* linear displacements. The linear displacements are shown in Fig. 11–22*b*, where *B* moves Δ_1 to B' and *C* moves Δ_3 to C'. These displacements cause the members' cords to rotate ψ_1, ψ_3 (clockwise) and $-\psi_2$ (counterclockwise) as shown.* Hence,

$$\psi_1 = \frac{\Delta_1}{3} \qquad \psi_2 = -\frac{\Delta_2}{3.6} \qquad \psi_3 = \frac{\Delta_3}{6}$$

As shown in Fig. 11–22*c*, the three displacements can be related. For example, $\Delta_2 = 0.5\Delta_1$ and $\Delta_3 = 0.866\Delta_1$. Thus, from the above equations we have

$$\psi_2 = -0.417\psi_1 \qquad \psi_3 = 0.433\psi_1$$

Using these results, the slope-deflection equations for the frame are

$$M_{AB} = 2E\left(\frac{I}{3}\right)[2(0) + \theta_B - 3\psi_1] + 0 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{3}\right)[2\theta_B + 0 - 3\psi_1] + 0 \qquad (2)$$

$$M_{BC} = 2E\left(\frac{I}{3.6}\right)[2\theta_B + \theta_C - 3(-0.417\psi_1)] - 32.4 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{3.6}\right)[2\theta_C + \theta_B - 3(-0.417\psi_1)] + 32.4 \qquad (4)$$

$$M_{CD} = 2E\left(\frac{I}{6}\right)[2\theta_C + 0 - 3(0.433\psi_1)] + 0 \qquad (5)$$

$$M_{DC} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_C - 3(0.433\psi_1)] + 0 \qquad (6)$$

$$M_{BA} + M_{BC} = 0 \quad (7)$$
$$M_{CD} + M_{CB} = 0 \quad (8)$$

$\circlearrowleft + \Sigma M_O = 0;$

$$M_{AB} + M_{DC} - \left(\frac{M_{AB} + M_{BA}}{3}\right)(10.2) - \left(\frac{M_{DC} + M_{CD}}{6}\right)(12.24) - 108(1.8)\ 0$$

$$-2.4M_{AB} - 3.4M_{BA} - 2.04M_{CD} - 1.04M_{DC} - 194.4 = 0 \quad (9)$$

$$0.733\theta_B + 0.167\theta_C - 0.392\psi_1 = \frac{9.72}{EI}$$

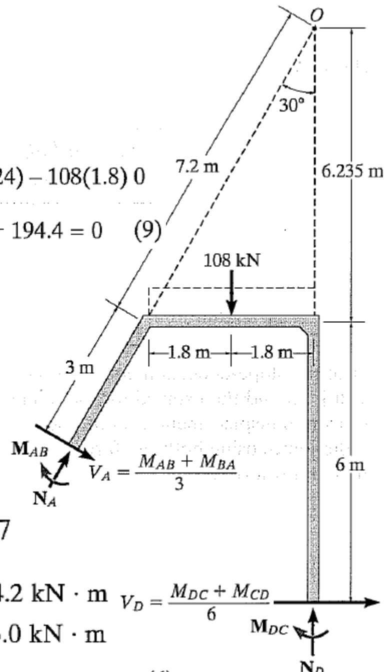
$$0.167\theta_B + 0.533\theta_C + 0.0784\psi_1 = -\frac{9.72}{EI}$$

$$-1.840\theta_B - 0.512\theta_C + 3.880\psi_1 = \frac{58.32}{EI}$$

$$EI\theta_B = 35.51 \qquad EI\theta_C = 33.33 \qquad EI\psi_1 = 27.47$$

$M_{AB} = -31.3\ \text{kN}\cdot\text{m}$ $M_{BC} = 7.60\ \text{kN}\cdot\text{m}$ $M_{CD} = -34.2\ \text{kN}\cdot\text{m}$

$M_{BA} = -7.60\ \text{kN}\cdot\text{m}$ $M_{CB} = 34.2\ \text{kN}\cdot\text{m}$ $M_{DC} = -23.0\ \text{kN}\cdot\text{m}$



(d)

November 22, 2023 Structural Analysis Prof. M. Samara 63