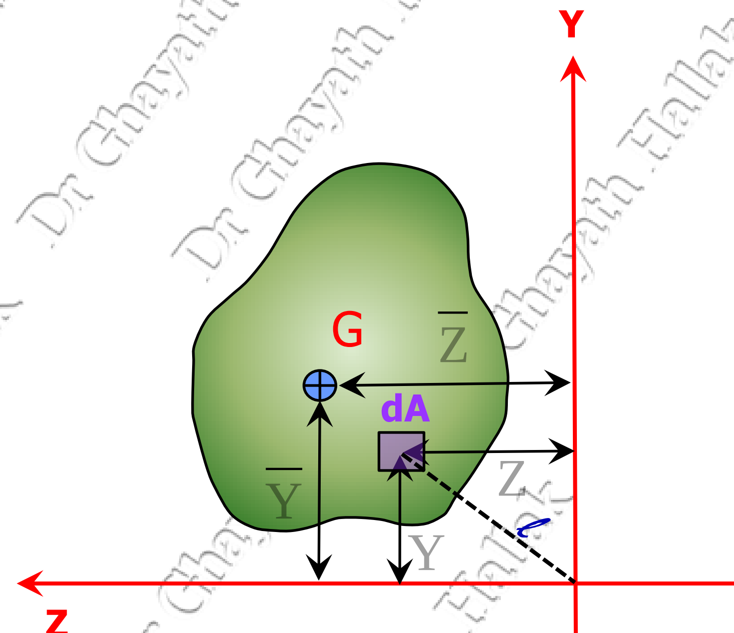
# MOMENT OF INERTIA
## The Second moment of area

# 1- The Second moment of area, The MOMENT OF INERTIA: ($mm^4$, $m^4$)

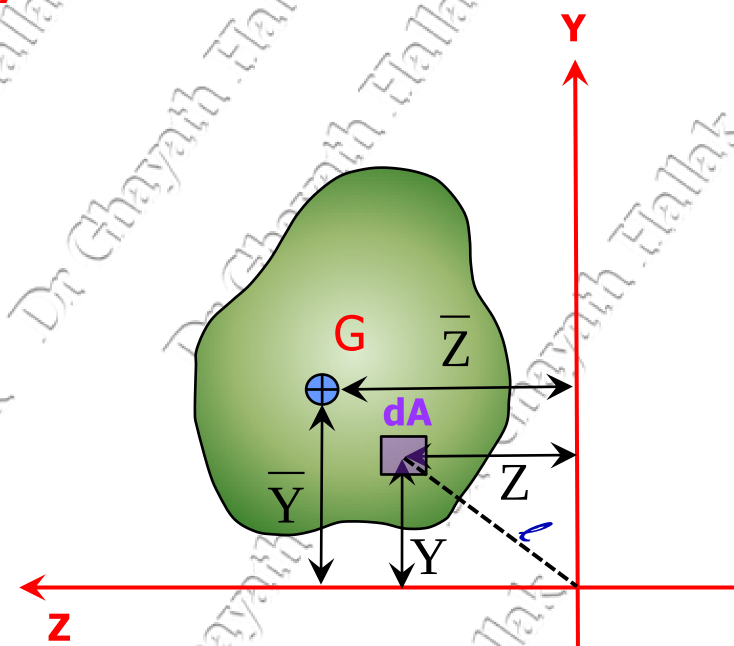
The **Moment of Inertia (I)** is a term used to describe the capacity of a cross-section to resist bending. It is always considered with respect to a reference axis such as Z or Y. It is a mathematical property of a section concerned with a surface area and how that area is distributed about the reference axis. The reference axis is usually a centroidal axis (NOT "Y & Z" axes shown in the Fig). The moment of Inertia expressed mathematically as:



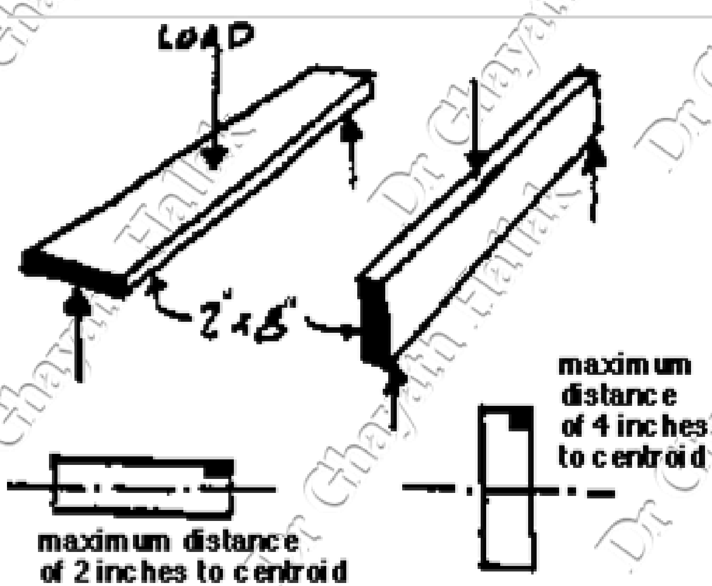

$$I_Z = \int_A Y^2\, dA \ , \ I_Y = \int_A Z^2\, dA$$

# 1- The Second moment of area, The MOMENT OF INERTIA: ($mm^4$, $m^4$)

The Moment of Inertia is an important value which is used to determine the state of stress in a section, to calculate the resistance to buckling, and to determine the amount of deflection in a beam.



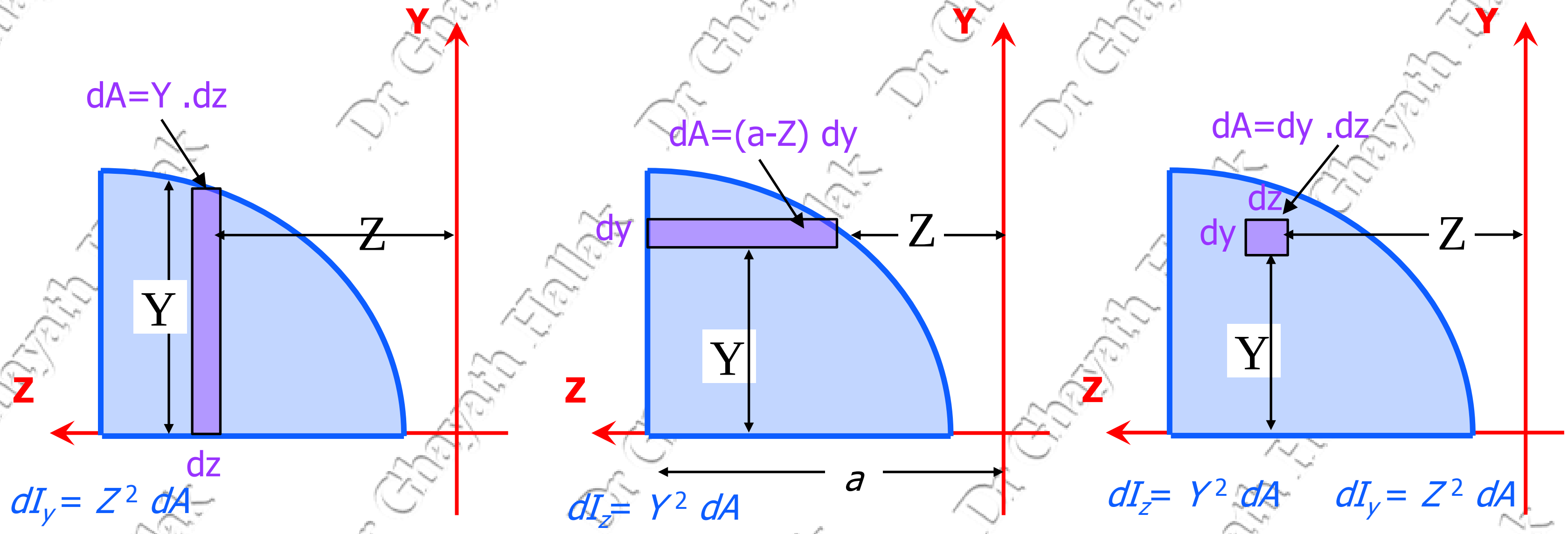

$$I_Z = \int_A Y^2\, dA \;,\; I_Y = \int_A Z^2\, dA$$




Both boards have the same cross-sectional area, but the area is distributed differently about the horizontal centroidal axis.

# 1- The Second moment of area, The MOMENT OF INERTIA: ($mm^4$, $m^4$)

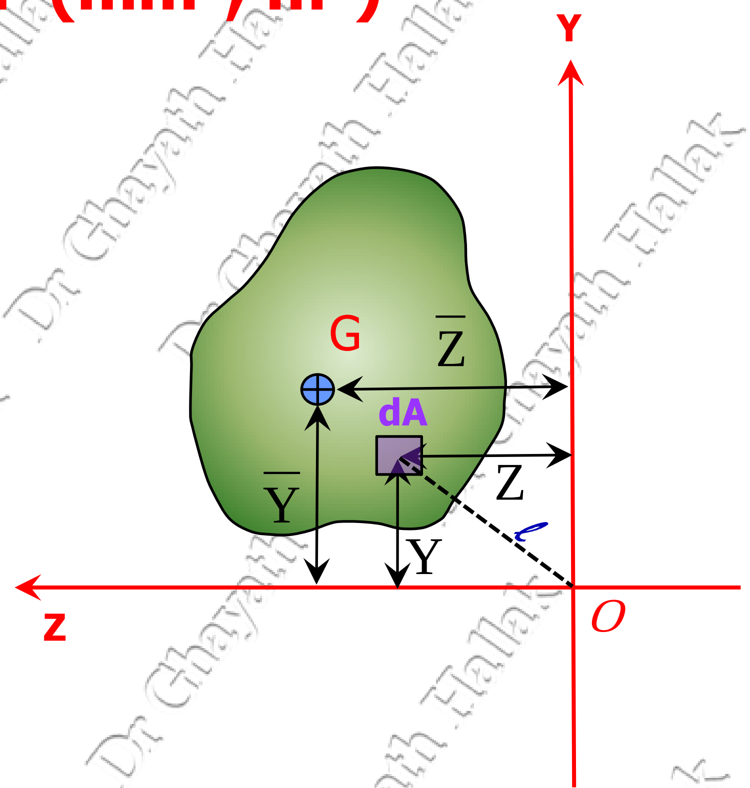
## DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION




**The moment of inertia of an area is always positive**

## 2- The polar moment of Inertia (mm$^4$, m$^4$)

Polar moment of inertia, denoted by $J_O$ or $I_p$, is the area moment of inertia about the $X$-axis (perpendicular to plan of cross-section area) given by:



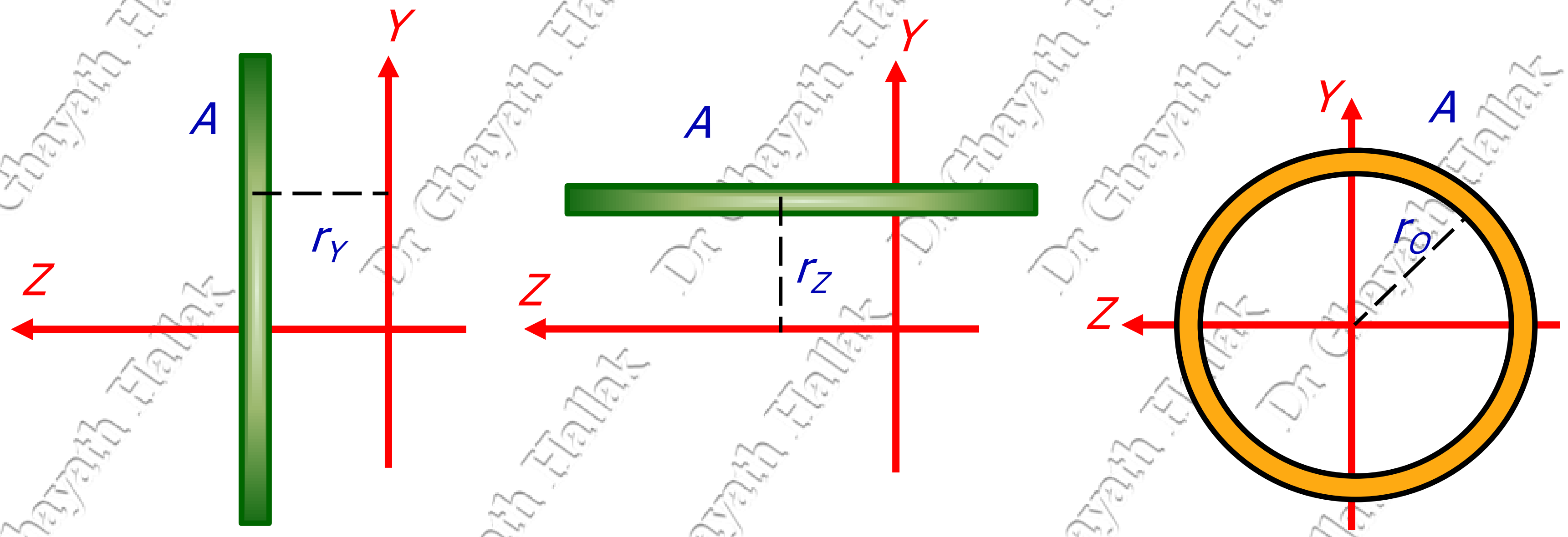

$$I_p = J_o = \int_A l^2\, dA = \int_A (Z^2 + Y^2)\, dA$$

$$= \int_A Z^2\, dA + \int_A Y^2\, dA$$

$$I_p = J_o = I_Y + I_Z$$

This integral of great importance in problems concerning the torsion of cylindrical shafts and in problems dealing with the rotation of slabs

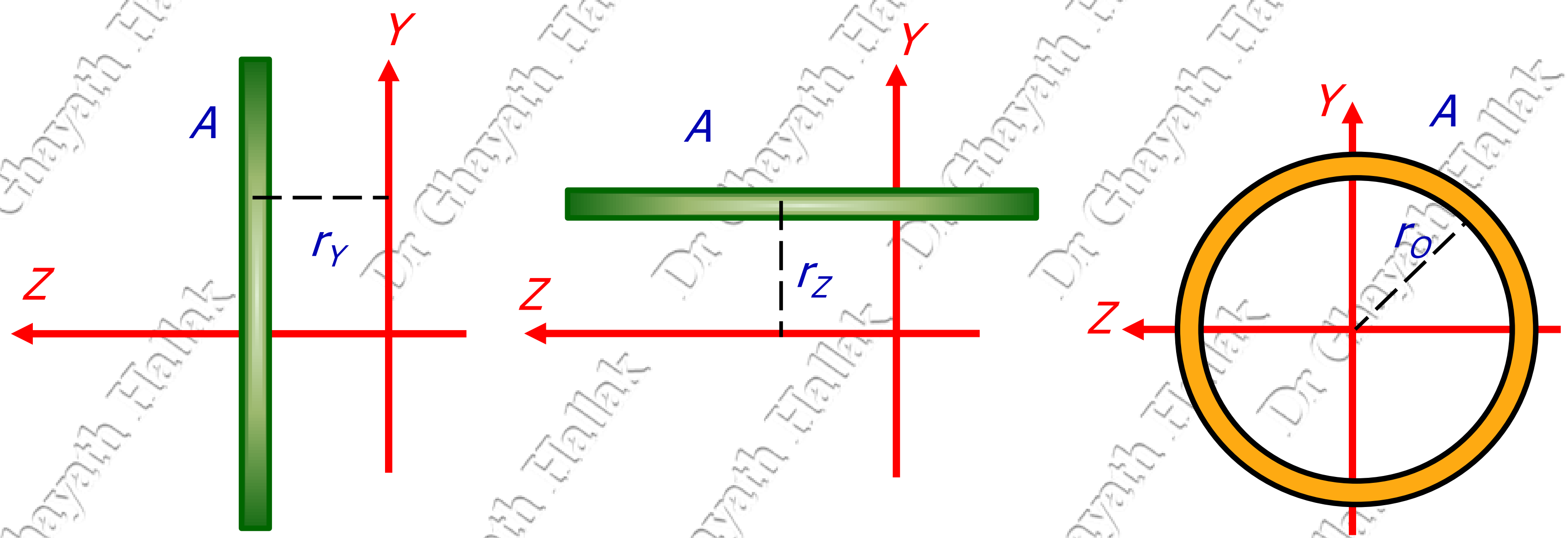
# 3- Radius of gyration (mm³, m³)




The radius of gyration is the distance *r* away from the axis that all the area can be concentrated to result in the same moment of inertia. That is,

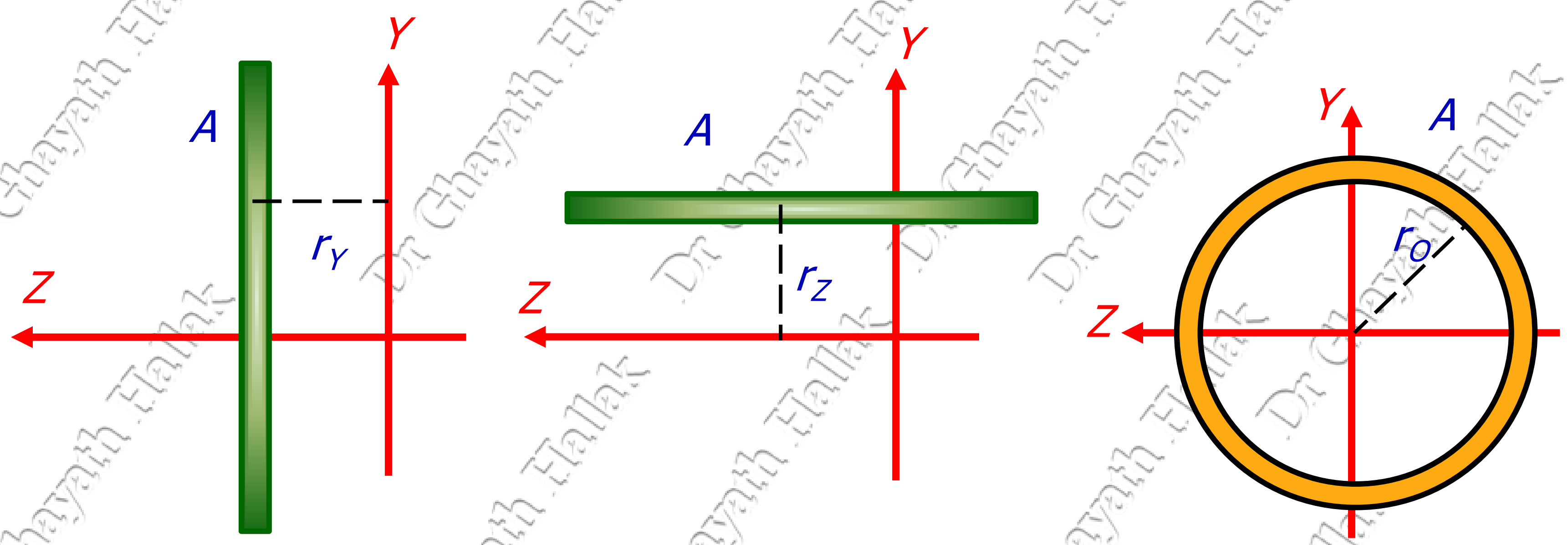
$$I = r^2 A = i^2 \, A$$

# 3- Radius of gyration (mm$^3$, m$^3$)




For a given area, one can define the radius of gyration around the *Y*-axis, denoted by $r_Y$ ,($i_Y$) the radius of gyration around the *Z*-axis, denoted by $r_Z$ , ($i_Z$) and the radius of gyration around the *X*-axis, denoted by $r_O$, ($i_O$). These are calculated from the relations:

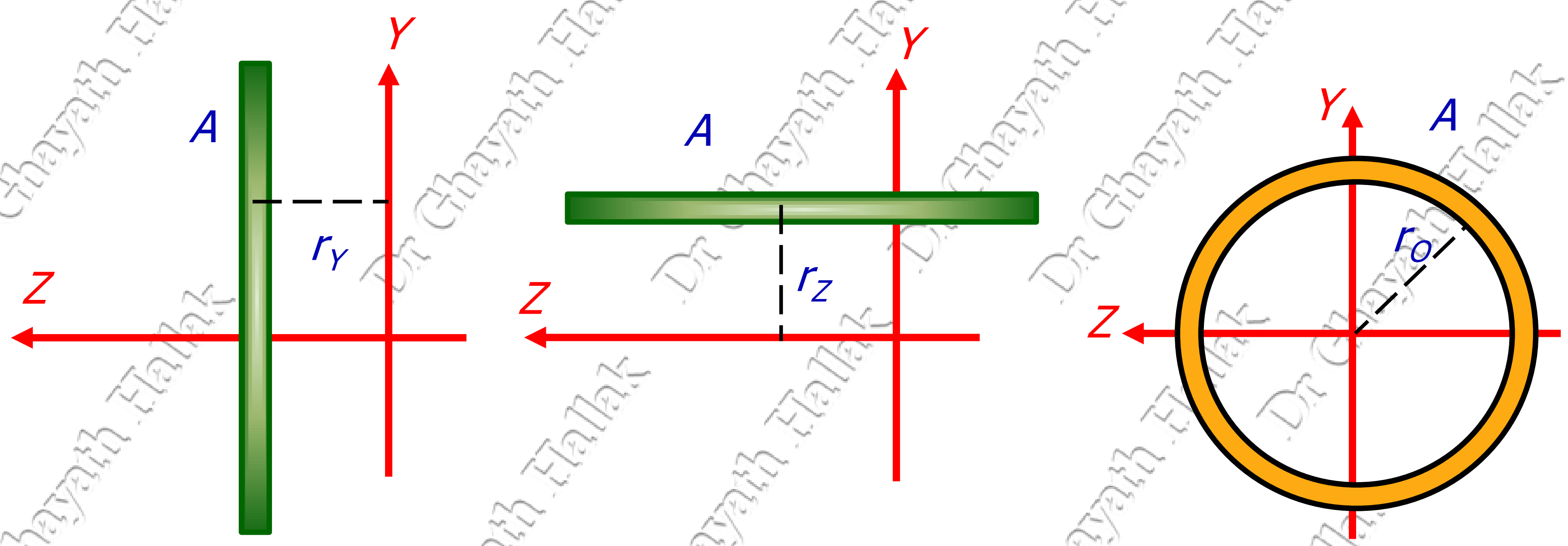
# 3- Radius of gyration (mm³, m³)




$$i_Z^2 = r_Z^2 = \frac{I_Z}{A}, \quad i_Y^2 = r_Y^2 = \frac{I_Y}{A}, \quad i_O^2 = r_O^2 = \frac{J_O}{A}$$

$$i_Z = r_Z = \sqrt{\frac{I_Z}{A}}, \quad i_Y = r_Y = \sqrt{\frac{I_Y}{A}}, \quad i_O = r_O = \sqrt{\frac{J_O}{A}}$$

# 3- Radius of gyration (mm³, m³)



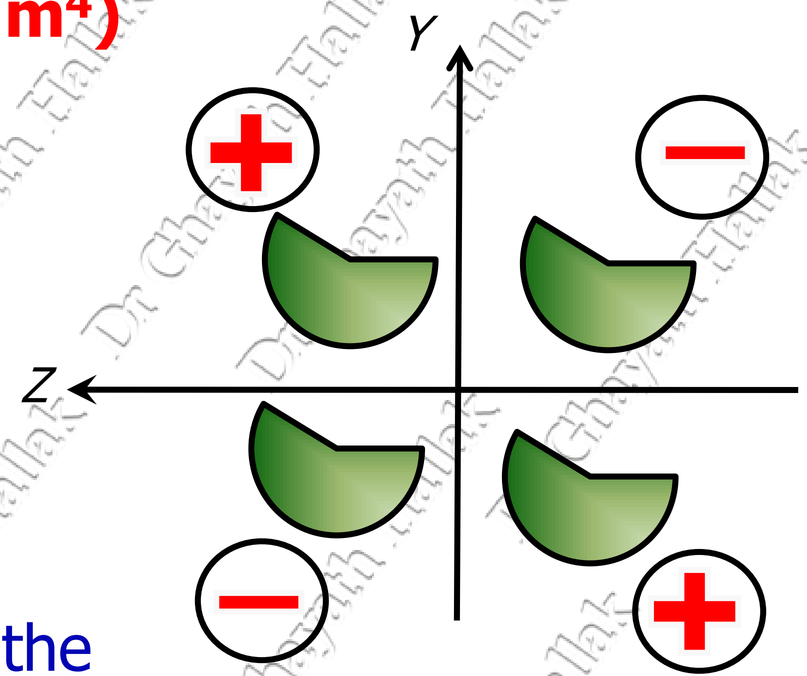

It can easily to show from $J_O = I_Y + I_Z$ that

$$r_Z^2 + r_Y^2 = r_O^2$$

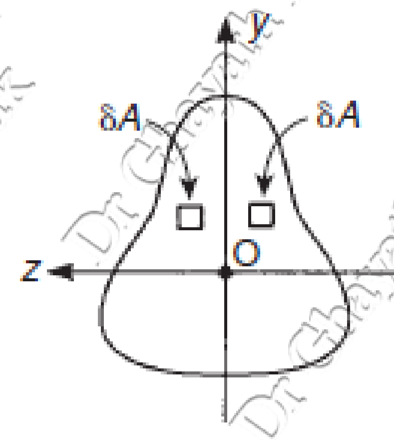
# 4- Product of Inertia (mm$^4$, m$^4$)

The product second moment of area, $I_{zy}$ , of a beam section with respect to z and y axes is defined by:

$$I_{ZY} = \int_A Z\,Y\,dA$$

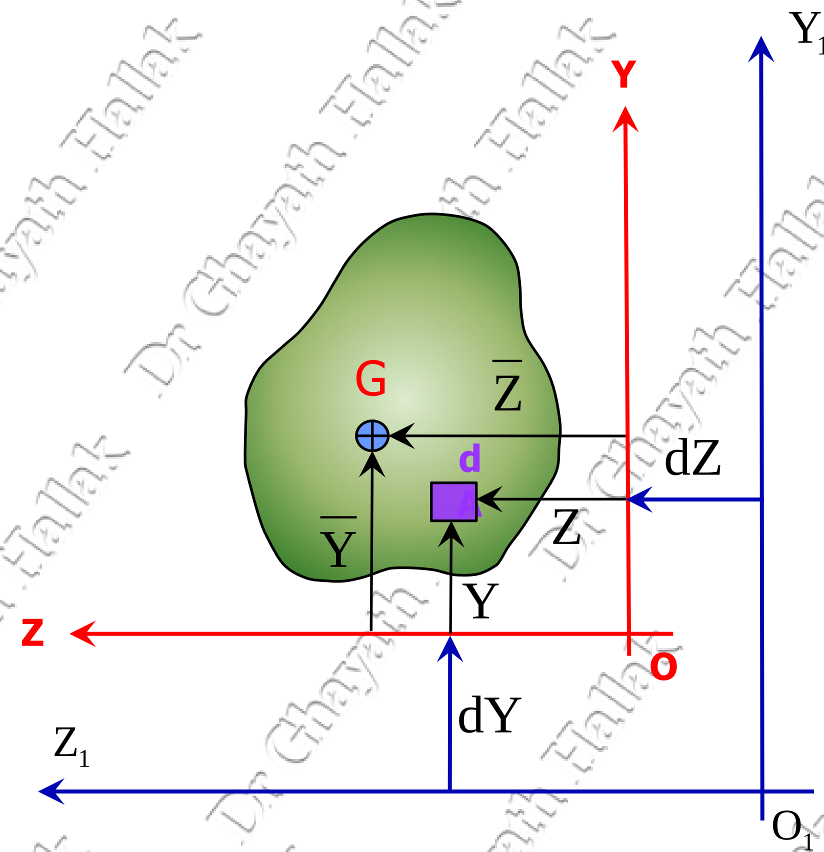



when one (or both) of the coordinate axes is an axis of symmetry → $I_{zy} = 0$



# 5- Parallel-Axis Theorems:

Suppose that we know the value of $I_Y$, $I_Z$ and $I_{ZY}$. We need to determine the value of $I_{Y1}$, $I_{Z1}$ and $I_{Y1Z1}$ (moment of Inertia according to the new axes Y1 and Z1 "parallel axis")




$$I_{Z1} = \int_A Y_1^2 dA$$

$$I_{Z1} = \int_A (Y + dY)^2 dA$$

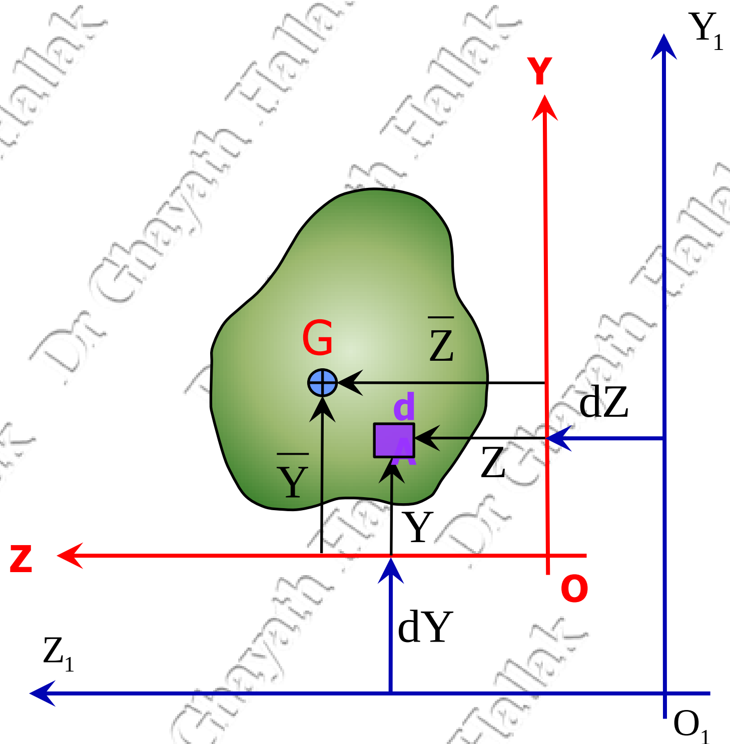
$$I_{Z1} = \int_A Y^2 dA + dY^2 \int_A dA + 2\, dY \int_A Y\, dA$$

# 5- Parallel-Axis Theorems:

$$I_{Z1} = I_Z + 2\ dY\ Q_Z + dY^2\ A$$

**Similarly**

$$I_{Y1} = I_Y + 2\ dZ\ Q_Y + dZ^2\ A$$




## Product of Inertia

$$I_{Z1Y1} = \int_A Y_1\ Z_1\ dA = \int_A (Y + dY)\,(Z + dZ)\,dA$$

$$I_{Z1Y1} = \int_A Y\ Z\ dA + dY\int_A Z\ dA + dZ\int_A Y\ dA + dY\ dZ\int_A dA$$

$$I_{Z1Y1} = I_{ZY} + dY\ Q_Y + dZ\ Q_Z + dY\ dZ\ A$$

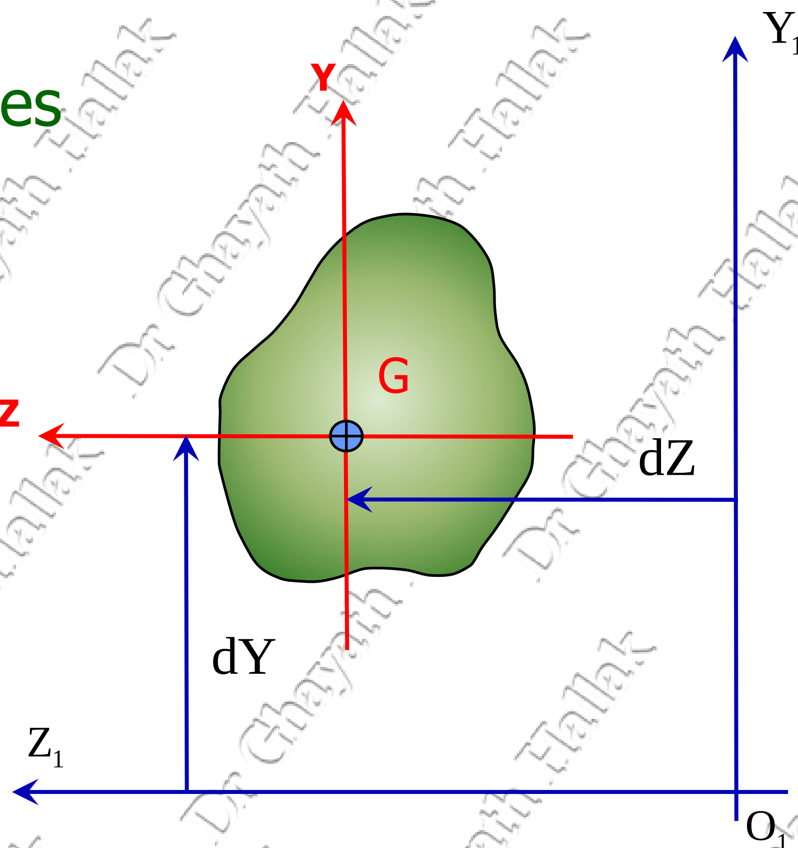
# 5- Parallel-Axis Theorems:

IF Z & Y are CENTROIDAL axes Then

$$I_{Z1} = I_{GZ} + dY^2\ A$$

$$I_{Y1} = I_{GY} + dZ^2\ A$$

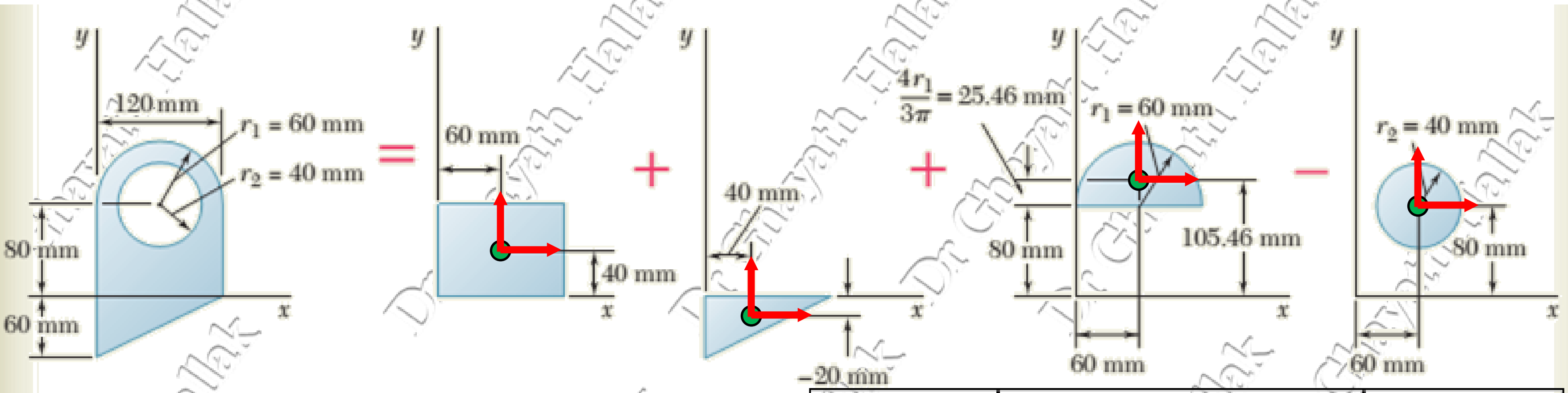
$$I_{Z1Y1} = I_{GZY} + dY\ dZ\ A$$

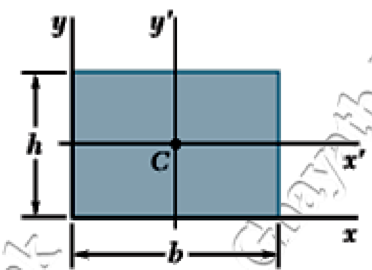
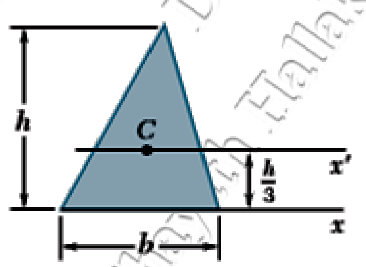


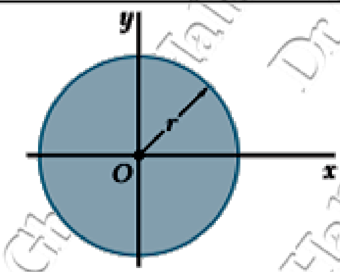
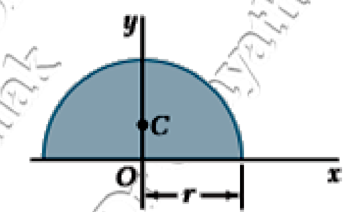
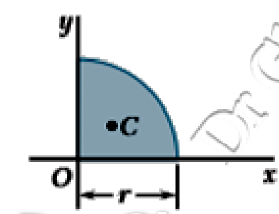

It can be seen from Eqs. above that if either G*Z or GY is an axis of symmetry, i.e.* $I_{GZY} = 0$, *then* $I_{Z1Y1} = dY\ dZ\ A$

Thus for a section component having an axis of symmetry that is parallel to either of the section reference axes the product second moment of area is the product of the coordinates of its centroid multiplied by its area.

# 5- MOMENTS OF INERTIA OF COMPOSITE AREAS




| | | |
|---|---|---|
| Rectangle |  | $\bar{I}_{x'} = \frac{1}{12}bh^3$<br>$\bar{I}_{y'} = \frac{1}{12}b^3h$<br>$I_x = \frac{1}{3}bh^3$<br>$I_y = \frac{1}{3}b^3h$<br>$J_C = \frac{1}{12}bh(b^2 + h^2)$ |
| Triangle |  | $\bar{I}_{x'} = \frac{1}{36}bh^3$<br>$I_x = \frac{1}{12}bh^3$ |

| | | |
|---|---|---|
| Circle |  | $\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$<br>$J_O = \frac{1}{2}\pi r^4$ |
| Semicircle |  | $I_x = I_y = \frac{1}{8}\pi r^4$<br>$J_O = \frac{1}{4}\pi r^4$ |
| Quarter circle |  | $I_x = I_y = \frac{1}{16}\pi r^4$<br>$J_O = \frac{1}{8}\pi r^4$ |

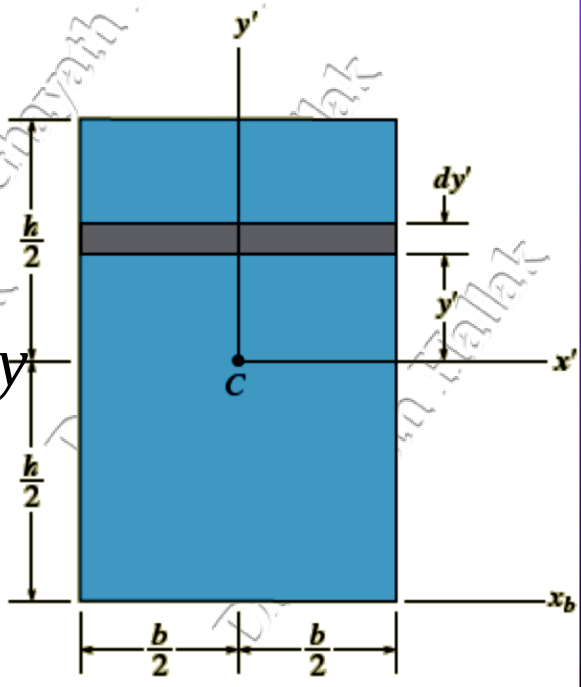
## EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

Determine the moment of inertia for the rectangular area with respect to (a) the centroidal x′ axis, (b) the centroidal x′ axis, (c) the axis $x_b$ passing through the base of the rectangular, and (d) the pole or z′ axis perpendicular to the x′-y′ plane and passing through the centroid C.

Part (a)

Differential element chosen, distance y′ from x′ axis.

Since dA = b dy′,




$$\bar{I}_x = \int_A y'^2\, dA = \int_{-h/2}^{h/2} y'^2 (b\, dy') = b\int_{-h/2}^{h/2} y'^2\, dy'$$

$$\bar{I}_x = b\left[\frac{y'^3}{3}\right]_{-h/2}^{+h/2} = \frac{1}{12} b h^3$$

# EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

Part (b)

Differential element chosen, distance x′ from y′ axis.

Since dA = h dx′,

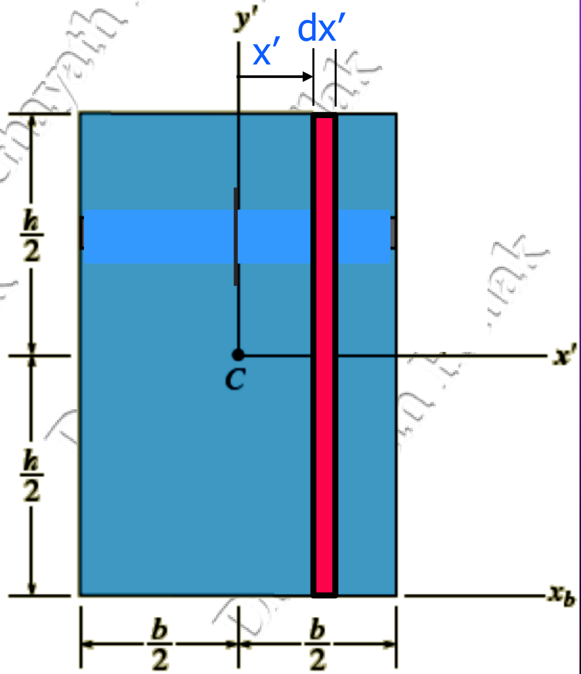
$$\bar{I}_y = \int_A x'^2 \, dA = \int_{-b/2}^{b/2} x'^2 (h dx') = h \int_{-b/2}^{b/2} x'^2 \, dy$$

$$\bar{I}_y = h\left[\frac{x'^3}{3}\right]_{-b/2}^{+b/2} = \frac{1}{12} hb^3$$

Part (c)

By applying parallel axis theorem,

$$I_{x_b} = \bar{I}_x + Ad^2 = \frac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3} bh^3$$



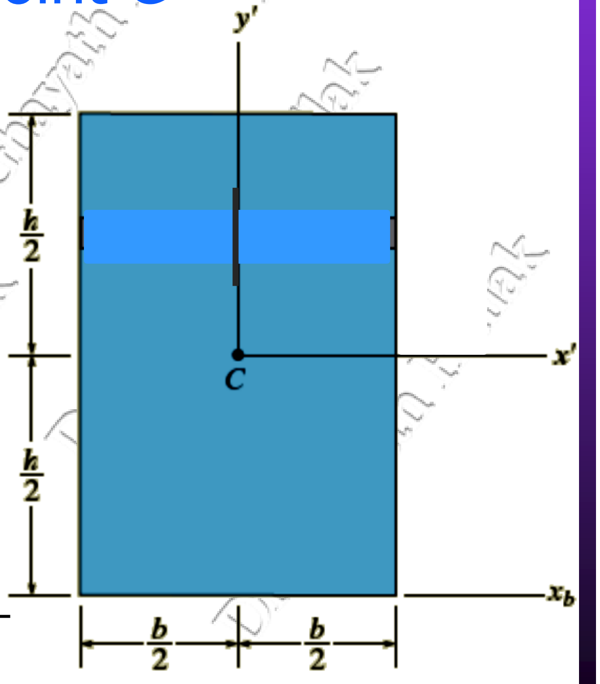
**EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION**

Part (d)

For polar moment of inertia about point C,

$$J_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12} bh(h^2 + b^2)$$

For radius of gyration about axis x′, y′ & point C



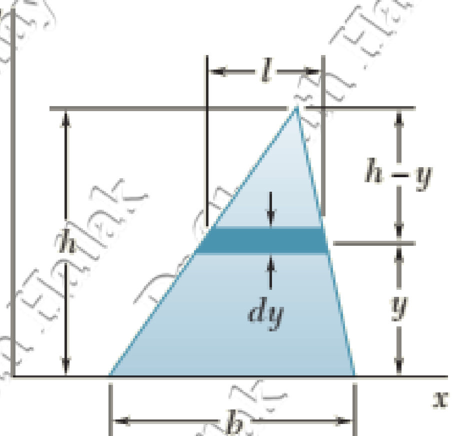

$$r_{x'} = \sqrt{\frac{\bar{I}_{x'}}{A}} = \sqrt{\frac{bh^3/12}{bh}} = \frac{h}{2\sqrt{3}}$$

$$r_{y'} = \sqrt{\frac{\bar{I}_{y'}}{A}} = \sqrt{\frac{hb^3/12}{bh}} = \frac{b}{2\sqrt{3}}$$

$$r_C^{\,2} = r_{z'}^{\,2} = r_{x'}^{\,2} + r_{y'}^{\,2} = \frac{h^2}{12} + \frac{b^2}{12} = \frac{h^2 + b^2}{12}$$

**EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION**

Determine the moment of inertia of a triangle with respect to its base.




$dI_x = y^2\, dA\,,\ dA = \ell\, dy$

Using similar triangles, we have

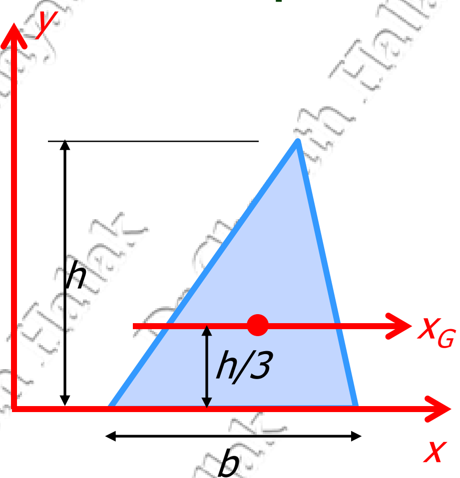
$$\frac{l}{b} = \frac{h-y}{h} \Rightarrow l = b\frac{h-y}{h},\therefore dA = b\frac{h-y}{h}dy$$

$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$I_x = \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h = \frac{bh^3}{12}$$

## EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

Determine the moment of inertia for the area with respect to $x_G$ axis




By applying parallel axis theorem,

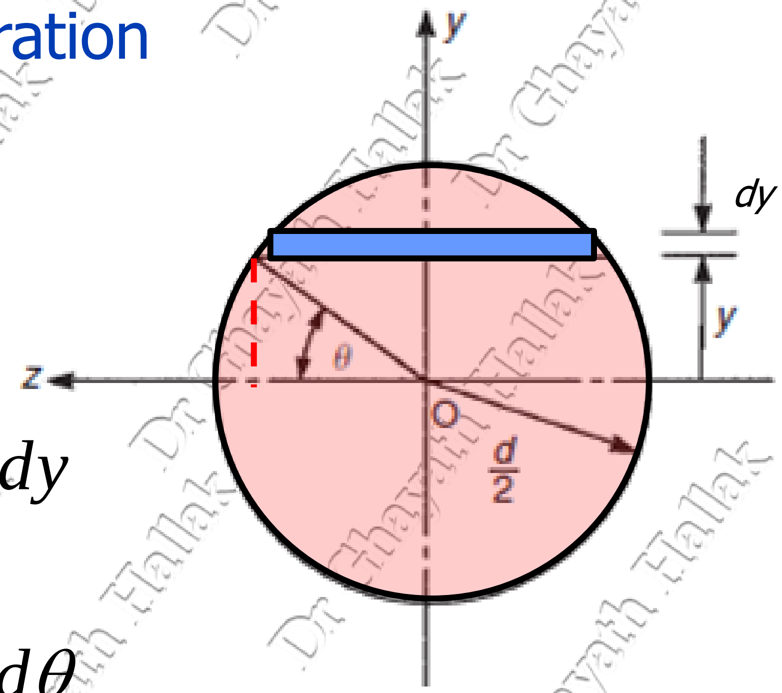
$$I_x = \bar{I}_{xG} + Ad^2 \Rightarrow \bar{I}_{xG} = I_x - Ad^2$$

$$\bar{I}_{xG} = \frac{1}{12}bh^3 - \frac{bh}{2}\left(\frac{h}{3}\right)^2 = \frac{1}{36}bh^3$$

## EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

(a) Determine the moment of inertia of a circular area with respect to a diameter.

(b) Determine the centroidal polar moment of inertia of a circular area by direct integration




$$dA = 2\left(\frac{d}{2}\cos\theta\right)dy$$

$$Iz = \int_A y^2 dA = \int_{-d/2}^{d/2} 2\left(\frac{d}{2}\cos\theta\right) y^2 dy$$

$$y = \left(\frac{d}{2}\sin\theta\right) \Rightarrow dy = \left(\frac{d}{2}\cos\theta\right)d\theta$$

$$y = \mp d/2 \Rightarrow \theta = \mp\pi/2$$

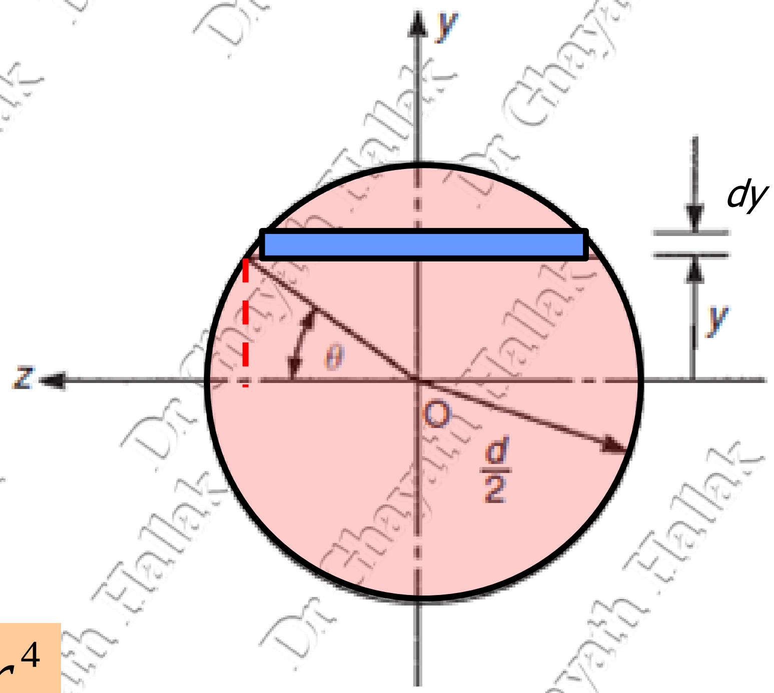
# EXAMPLS- DETERMINATION OF THE MOMENT OF INERTIA OF AN AREA BY INTEGRATION

$$Iz = \int_{-\pi/2}^{\pi/2} (d\cos\theta)\left(\frac{d}{2}\sin\theta\right)^2\left(\frac{d}{2}\cos\theta\right) d\theta$$

$$Iz = \frac{d^4}{8}\int_{-\pi/2}^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta$$

$$\Rightarrow Iz = \frac{\pi d^4}{64} = Iy$$

$$\Rightarrow Iz = \frac{\pi r^4}{4} = Iy$$




$$Jo = Iz + Iy = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}$$