جــامـــعـــة دمـــشـــق
كـلـيـة الـهـنـدسـة المـدنيـة

تـصـمـيــم بـمـعـونــة الـحــاســب
Computer Aided Design

Lecture 5

Optimization & Mathematical Programming

9/12/2025

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Last Lecture

- Statistics
  - Graphics
- Regression and Curve Fitting
  - Linear
  - Higher order
  - Example
- Optimization
- Projectile Motion
  - Analytical solution
  - Numerical solution
- Other Examples
- Layout Optimization
  - http://www.layopt.com/

Content

- Mathematical Programming
  - Linear Programming
- Optimization Examples

Mathematical Programming

- Variables
  - Continuous, e.g. geometry
  - Discrete, e.g. section or material selection (integer, binary)
- Constraints
  - Linear vs. nonlinear
  - Equality vs. inequality
  - Objective function
    - Maximization, minimization
- Linear programming

Linear Programming

- Variables (n):

$$x_1, x_2, \cdots, x_n$$

- Minimize:

$$f(x_1, x_2, \cdots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

- Constraints (m):

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

$$x_1, x_2, \cdots, x_n \geq 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$a = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Linear Programming

- Variables (n):

$$x$$

- Minimize:

$$f(x) = c^T x$$

- Constraints (m):

$$ax = b$$

$$x \geq 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$a = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Linear Programming

- Minimization problem
- Maximize

$$f(x) = c^T x$$

Equivalent to minimize

$$-f(x) = -c^T x$$

- Objective value

Equivalent to minimizing error

Linear Programming

- Constraints:
  - Equality:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

  - Inequality constraints:

  - $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$

  Equivalent to $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + x_{n+1} = b_i$

  - $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \ge b_i$

  Equivalent to $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - x_{n+1} = b_i$

Linear Programming

- Constraints:

$$x_i \geq 0$$

- For variables that could be negative:

$$x_i = x_i^+ - x_i^-$$

$$x_i^+, x_i^- \geq 0$$

Linear Programming

- Solution:
  - Solution vs. no solution
  - Single vs. multiple solutions
  - Bounded vs. unbounded solution

Linear Programming

Maximize:

$$z = c_1x + c_2y$$

Subject to:

$$a_1x + b_1y \leq d_1$$

$$a_2x + b_2y \leq d_2$$

$$x, y \geq 0$$

Maximize:

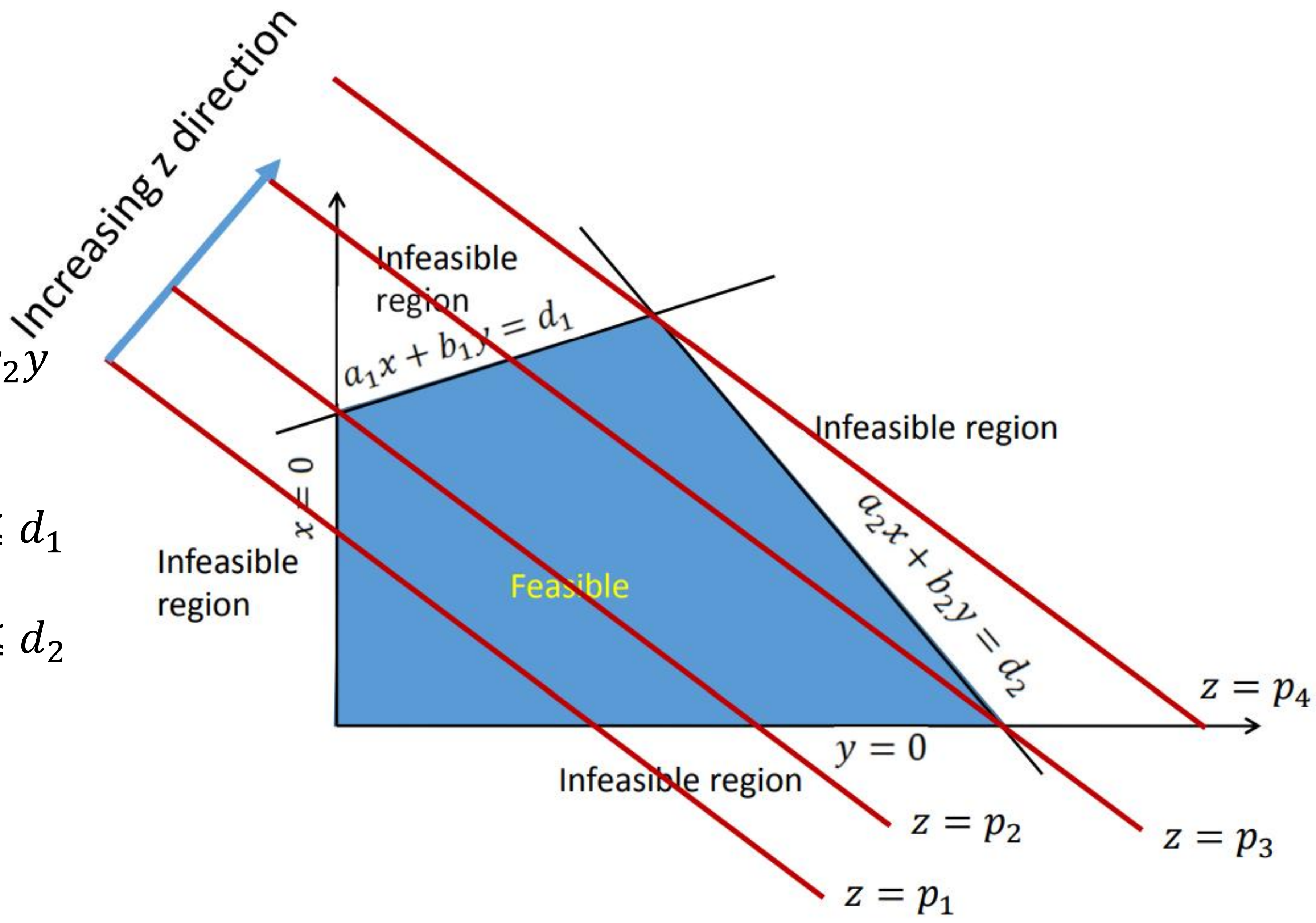
$$z = c_1 x + c_2 y$$

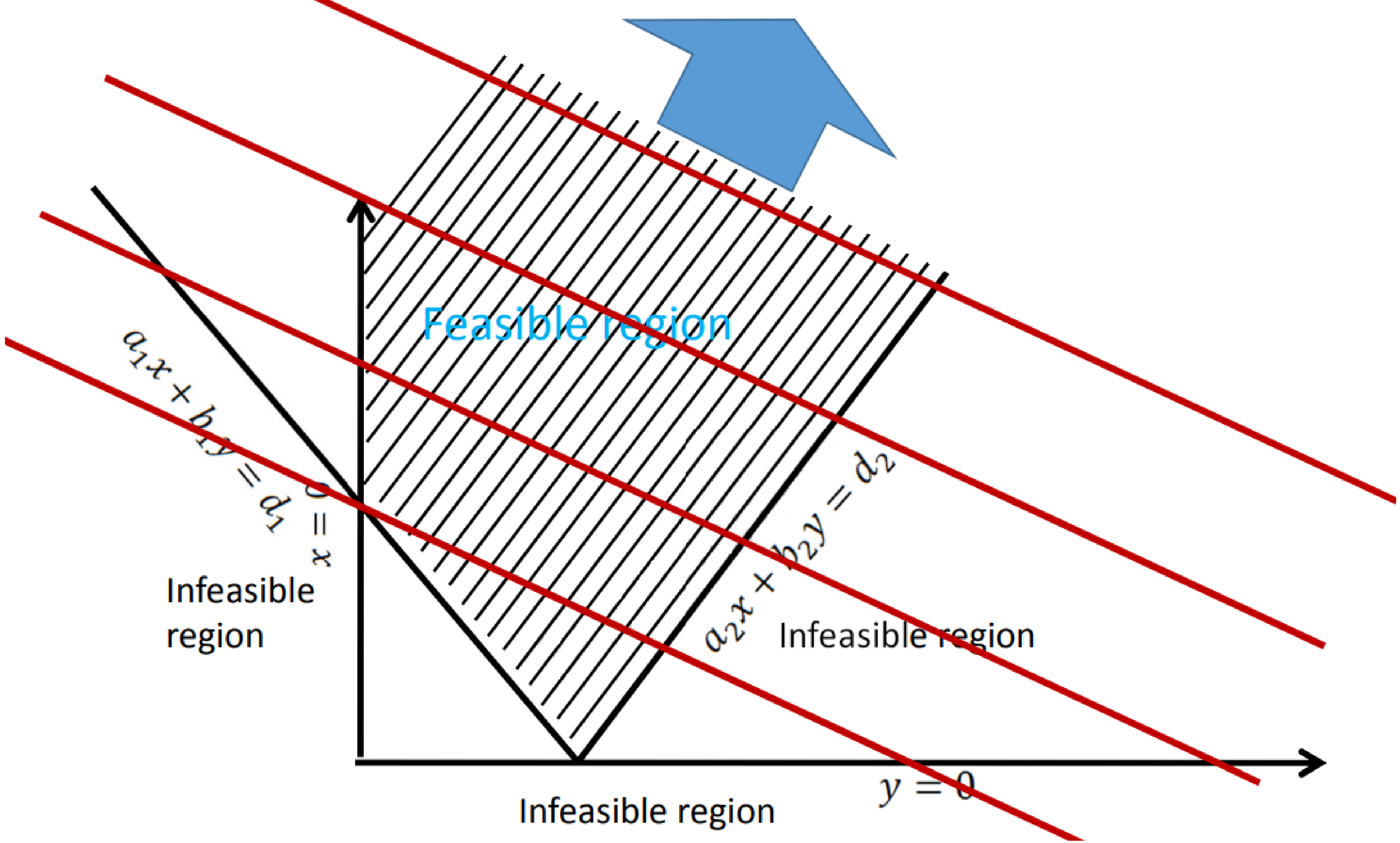
Subject to:

$$a_1 x + b_1 y \leq d_1$$

$$a_2 x + b_2 y \leq d_2$$

$$x, y \geq 0$$




Feasible region
$a_1x + b_1y = d_1$
$x = 0$
Infeasible region
$a_2x + b_2y = d_2$
Infeasible region
$y = 0$
Infeasible region

Mathematical Programming

- Numerical examples

Example 1

- Maximize $f(x,y) = -x^2 + 10.x - y^2 + 8.y$

- Analytically:

$$\frac{\delta f(x,y)}{\delta x} = -2.x + 10 = 0$$

$$\frac{\delta f(x,y)}{\delta y} = -2.y + 8 = 0$$

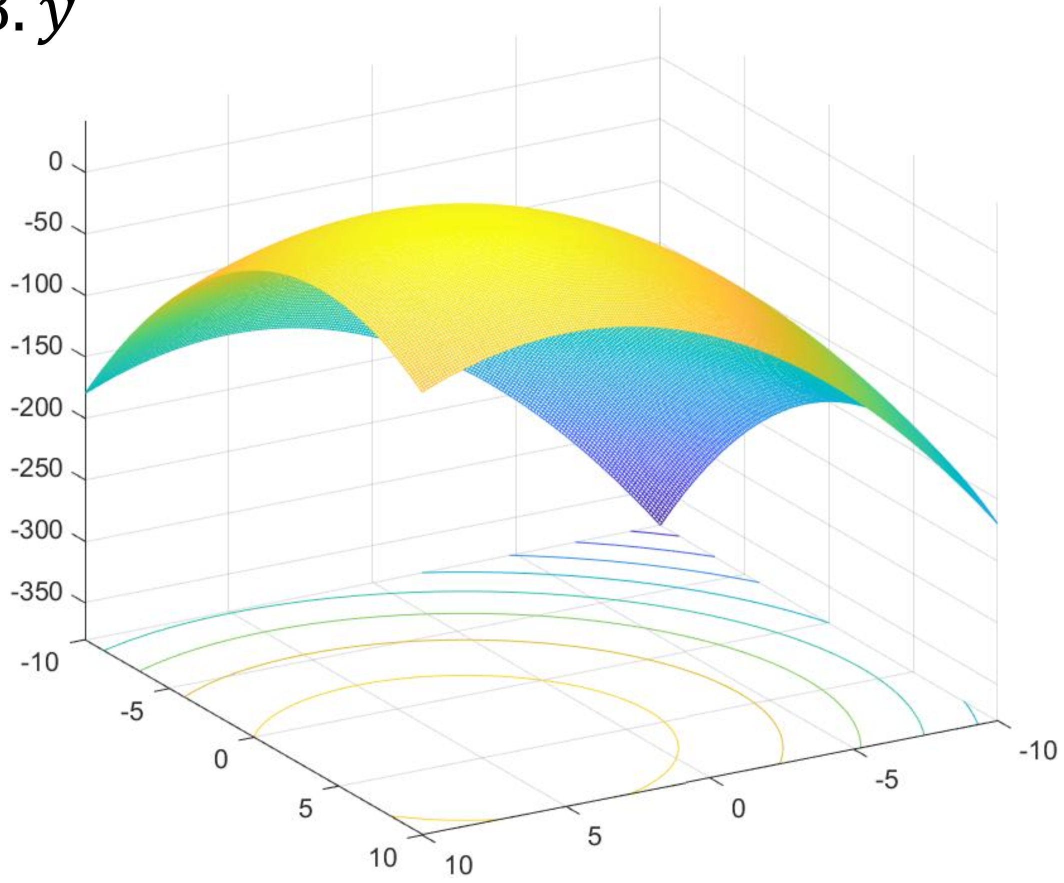
$$x = 5$$
$$y = 4$$
$$f(x,y) = 41$$

Example 1

- Maximize $f(x, y) = -x^2 + 10.x - y^2 + 8.y$

- Numerically:

```
[x, y] = meshgrid(0:0.1:10);
f = -x.^2 + 10*x - y.^2 + 8*y;
meshc(x, y, f)
max(max(f))  % 41
```



Example 1

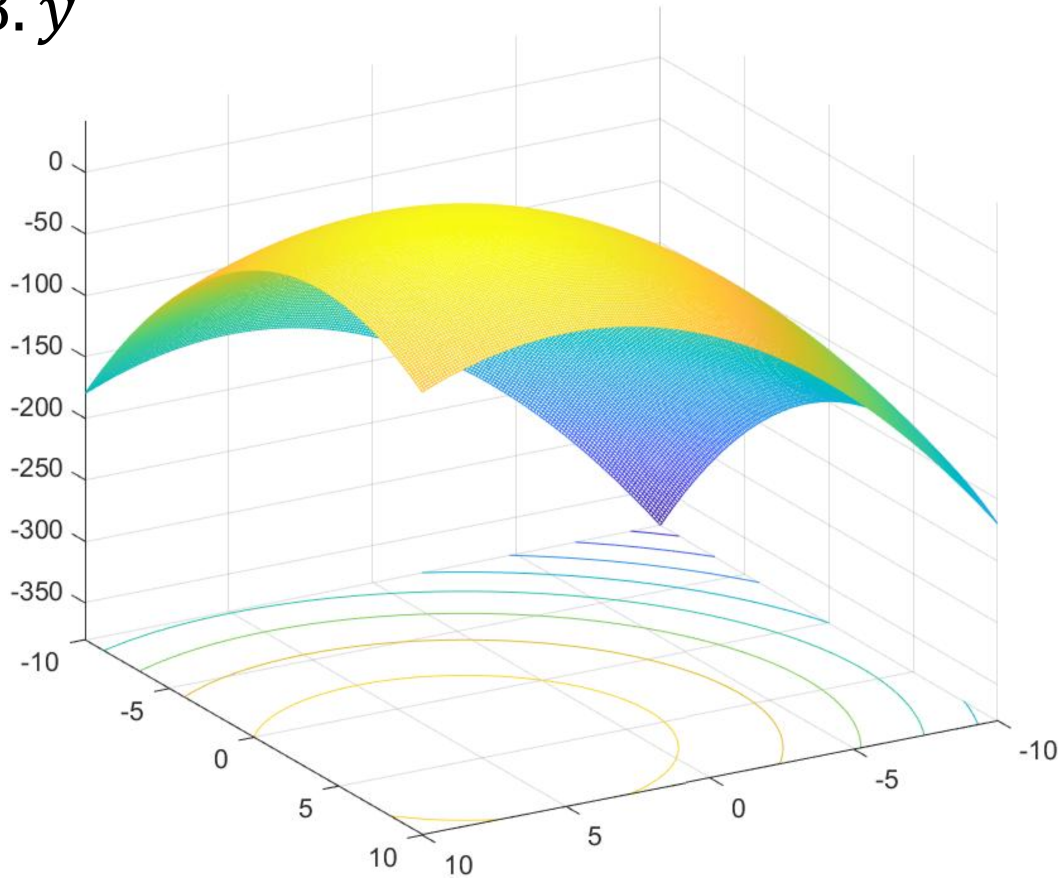
- Maximize $f(x, y) = -x^2 + 10.x - y^2 + 8.y$

- Symbolic:

```
f = @(x)  -x(1).^2+10*x(1)-x(2).^2+8*x(2);
```

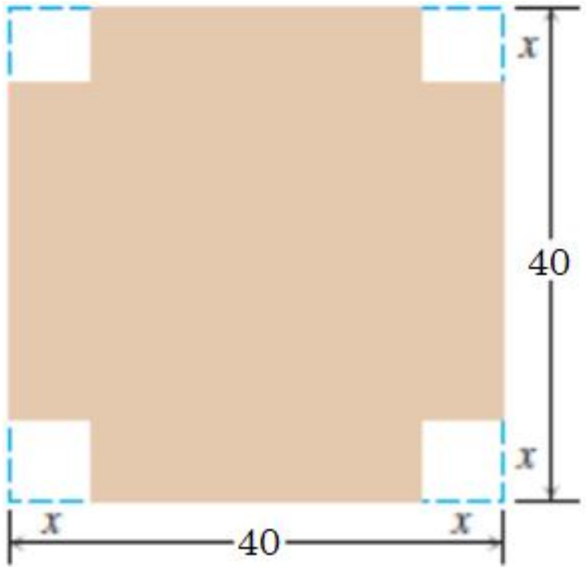
Minimization only

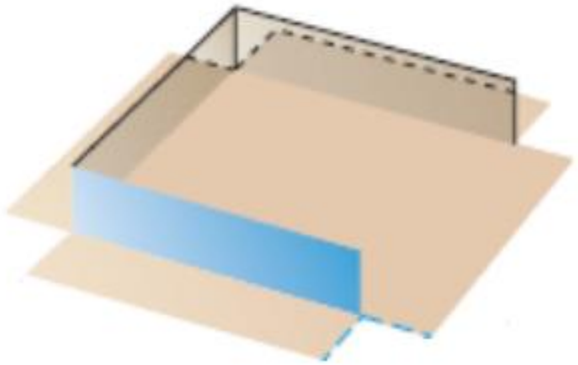
```
f = @(x)-(-x(1).^2+10*x(1)-x(2).^2+8*x(2));
x0 = [0, 0];
sol = fminsearch(f, x0);
disp(sol) % 5, 4
f(sol)    % -41
```



Example 2

- An open rectangular box with square base is made from 40x40cm sheet bended up as shown
- What is the largest possible volume?





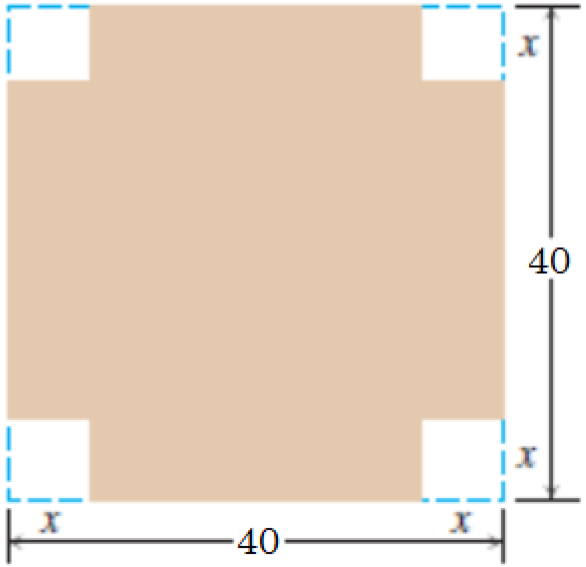
Example 2

- Variable x
- Objective volume:

$$v(x) = x.(40 - 2.x)^2 = 1600x - 160x^2 + 4x^3$$

- Subject to constraint:

$$0 < x < 20$$



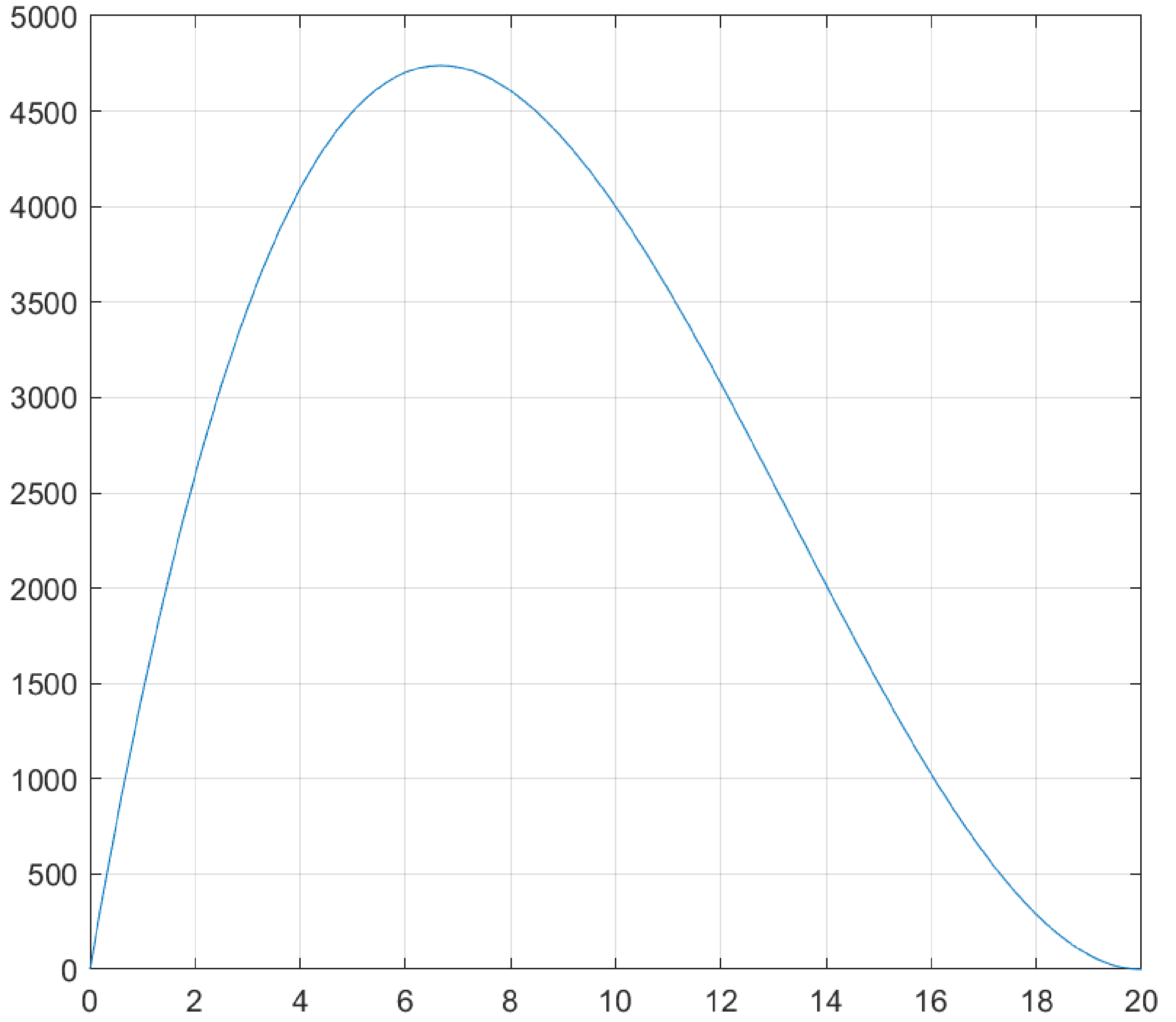
$$\frac{\delta v(x)}{\delta x} = 1600 - 320x + 12x^2 = 0$$

$$133.33 - 26.67x + x^2 = (x - 20).(x - 6.67) = 0$$

Example 2

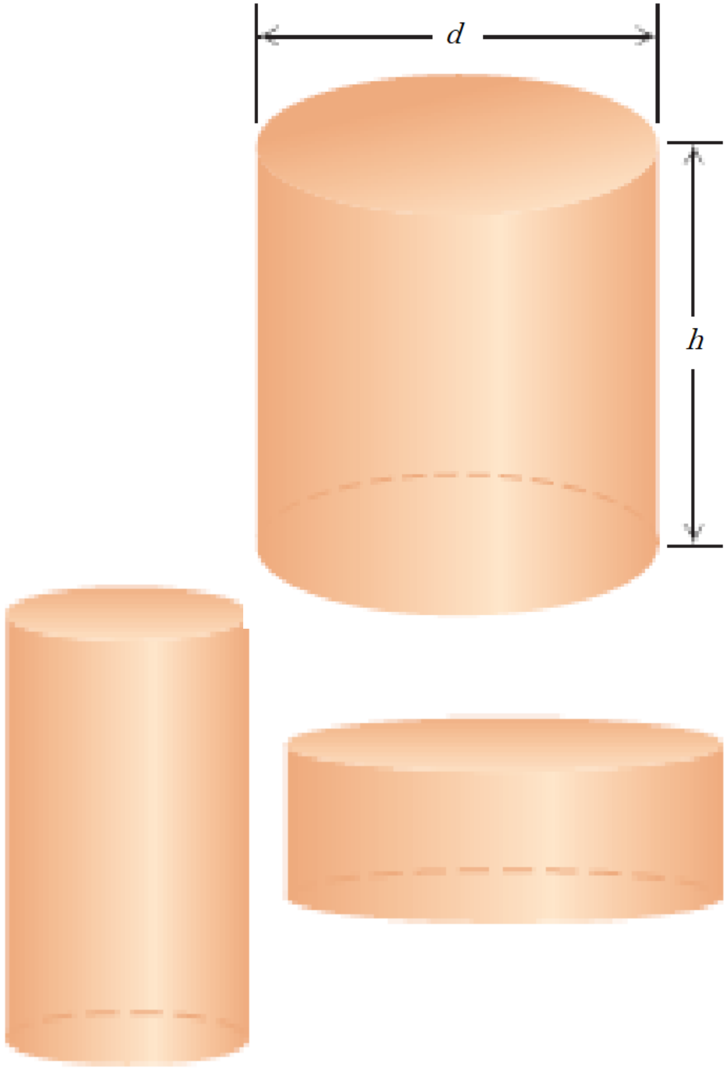
- $v(x) = 1600x - 160x^2 + 4x^3$
- $1600 - 320x + 12x^2 = 0$
  - $x = 20$
  - $x = 6.67$
- Objective volume $= 4740.7$
- Numerical:

```
x = linspace(0, 20);
v = 1600.*x-160.*x.^2+4*x.^3;
plot(x, v), grid
roots([12 -320 1600])
```



Example 3

- Design optimum cylindrical tank with minimum material and volume 1000 litre
- Variables *d* and *h*
- Minimize surface area
- Subject to volume



Example 3

- Variables d, h in metre
- Objective surface:

$$a(d,h) = \pi.d.h + 2 \times 0.25\pi d^2 = \pi d(h + 0.5d)$$

- Subject to constraint:

$$v(d,h) = 0.25\pi d^2 h = 1 \;\;\Rightarrow\;\; h = \frac{4}{\pi d^2}$$

$$a(d,h) = \pi d\left(\frac{4}{\pi d^2} + 0.5d\right) = \frac{4}{d} + 0.5\pi d^2$$

$$\frac{\delta a(d)}{\delta d} = \pi d - \frac{4}{d^2} = 0 \;\Rightarrow d = \sqrt[3]{\frac{4}{\pi}} = 1.0839$$

Example 3

- Variables d, h in metre
- Objective surface:

$$a(d,h) = \pi . d . h + 2 \times 0.25\pi d^2 = \pi d(h + 0.5d)$$

- Subject to constraint:

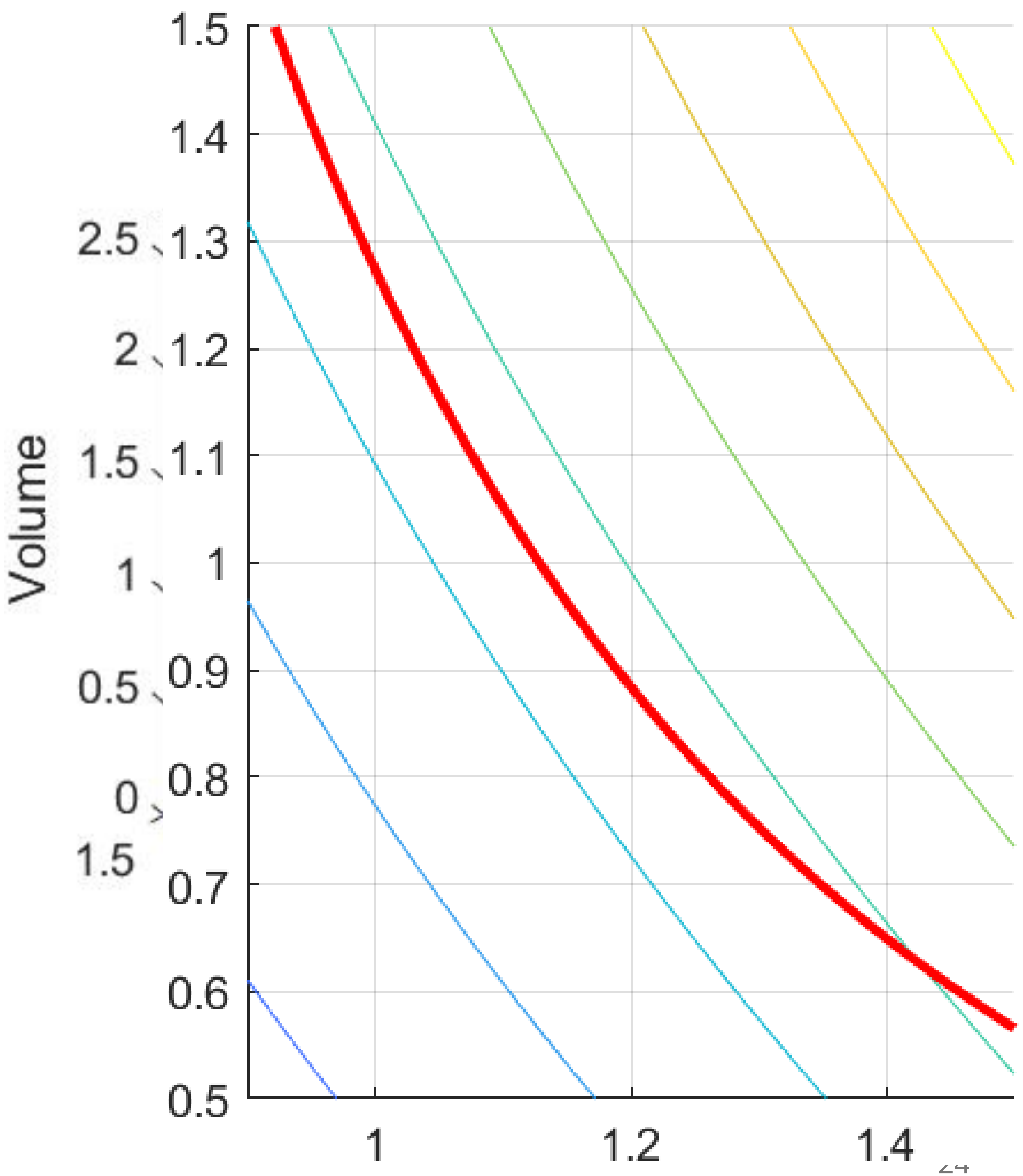
$$v(d,h) = 0.25\pi d^2 h = 1$$

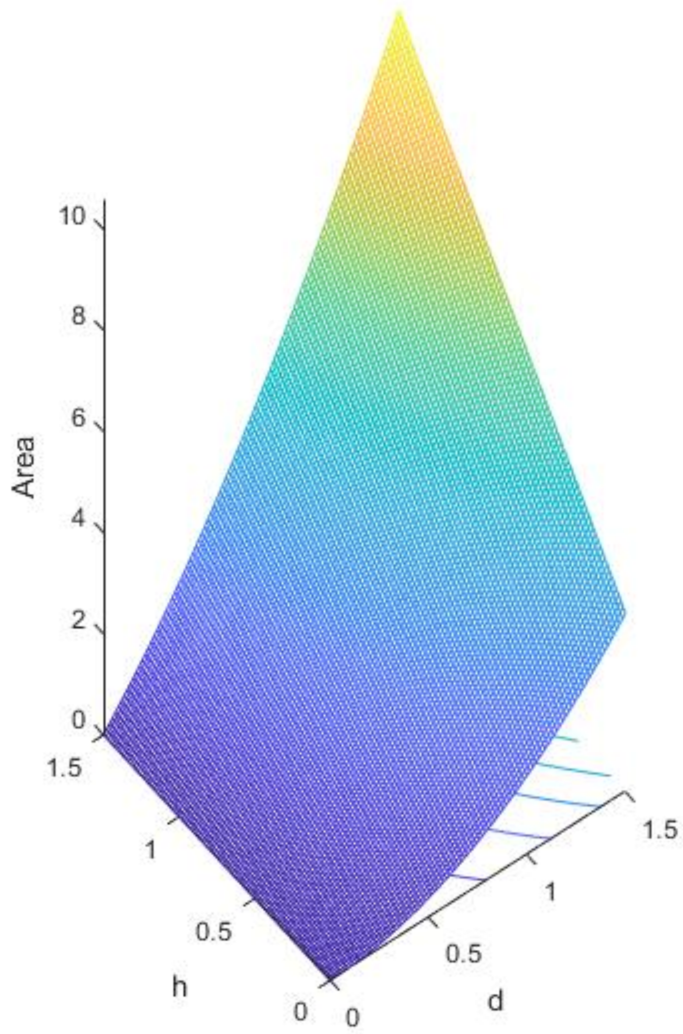
$$d = 1.0839$$

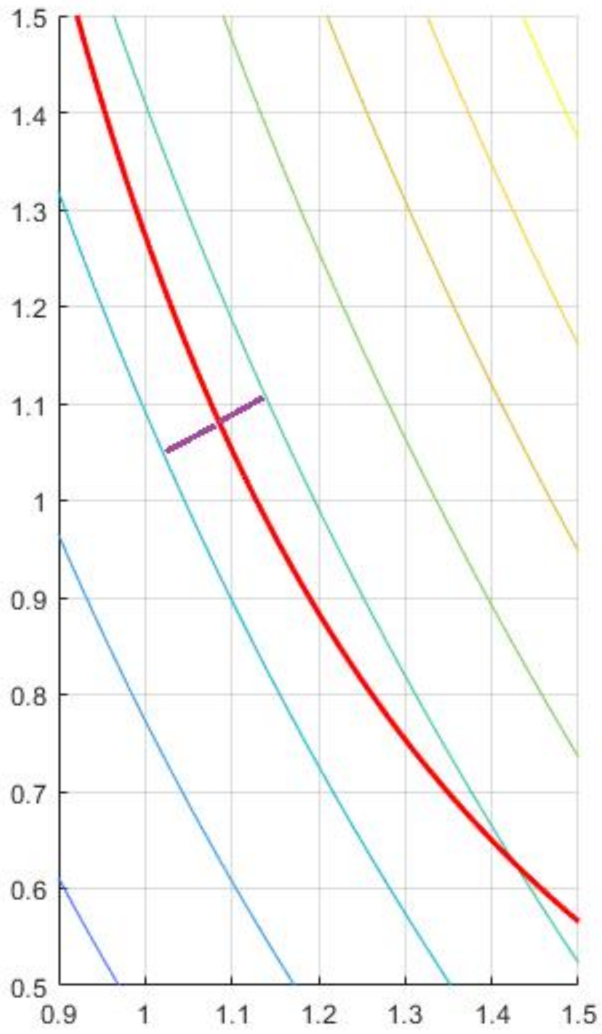
$$h = \frac{4}{\pi d^2} = 1.0839$$

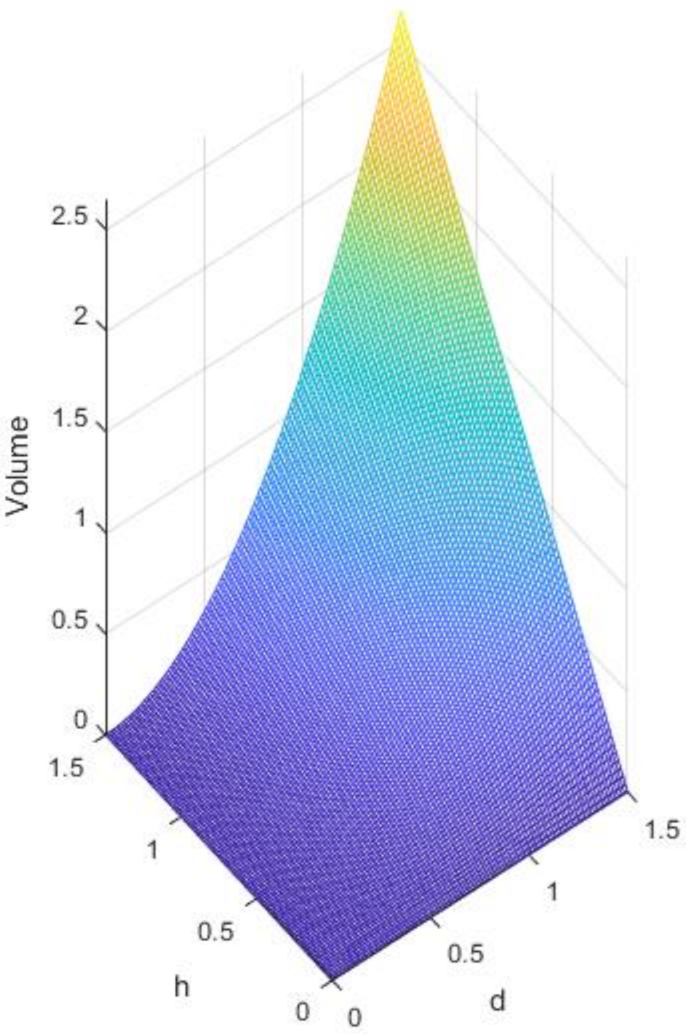
Example 3

```
d1 = linspace(0, 1.5);
h1 = linspace(0, 1.5);
[d, h] = meshgrid(d1, h1);

a = pi*d.*(h+0.5*d);
v = 0.25*pi*d.^2.*h;

meshc(d, h, a)
meshc(d, h, v)


contour(d, h, v, [1 1], ...
   LineWidth=2, EdgeColor='red')
contour(d, h, a)
```




Area
10
8
6
4
2
0
1.5
1
0.5
0
h
0
0.5
1
1.5
d


1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.9
1
1.1
1.2
1.3
1.4
1.5


Volume
2.5
2
1.5
1
0.5
0
1.5
1
0.5
0
h
0
0.5
1
1.5
d

Example 4

- What is the shortest fire truck ladder to get to the building?

Example 4

- Variables x & y
- Objective minimize L:

$$L^2 = y^2 + (x+4)^2$$

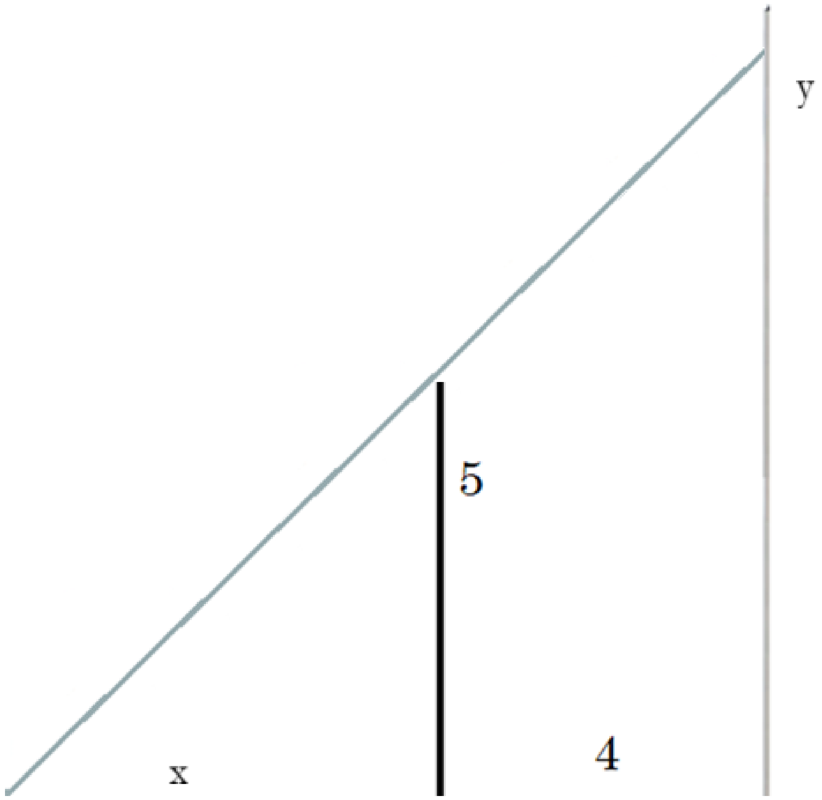
- Constraint:

$$0 < x < \infty$$

$$\tan(\theta) = \frac{y}{4+x} = \frac{5}{x}$$

$$y = 5(4+x)/x$$

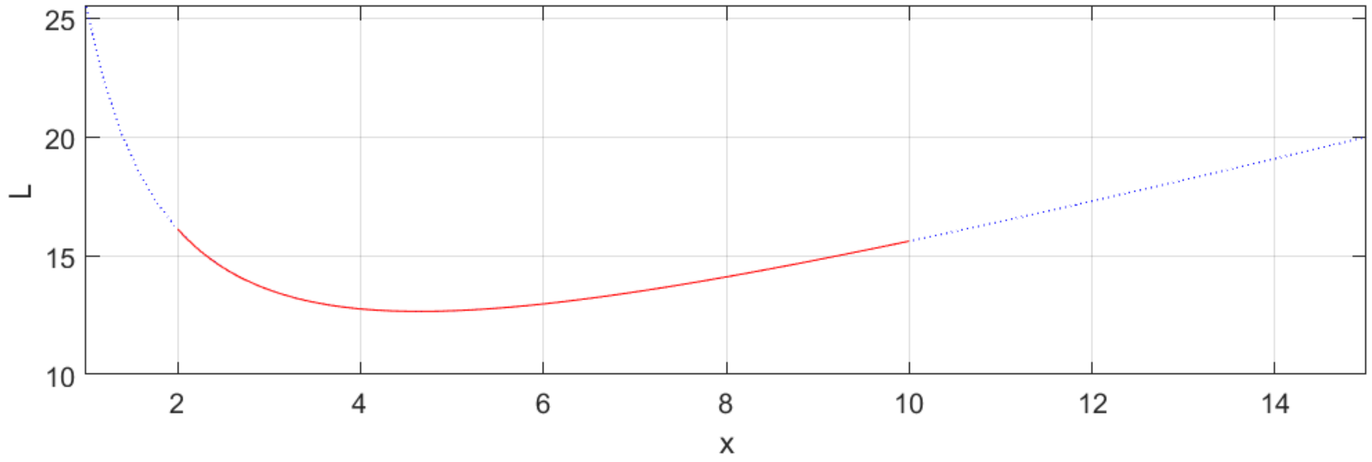
$$L^2 = (\frac{5(4+x)}{x})^2 + (x+4)^2$$



Example 4

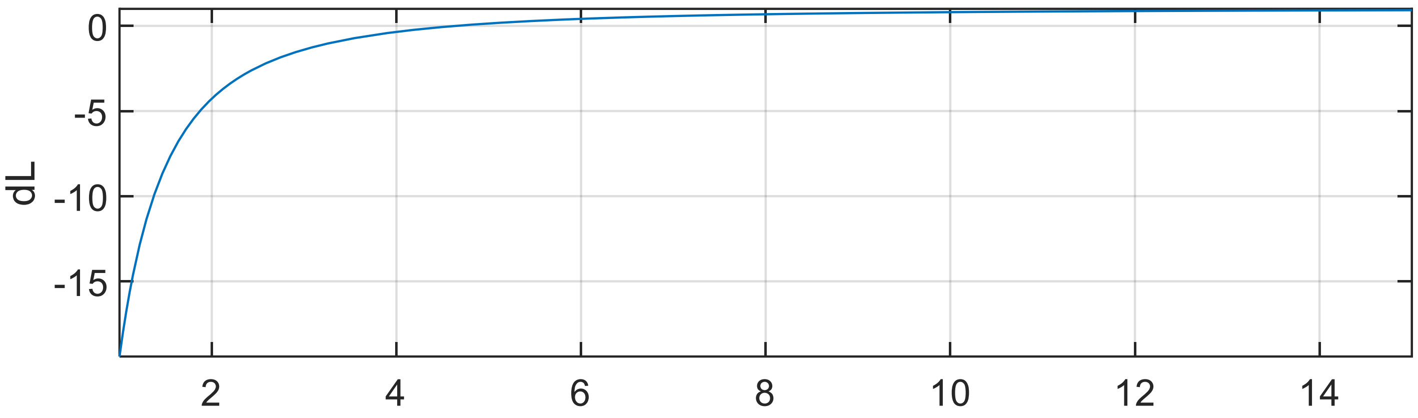
$$L = \sqrt{\left(\frac{5(4+x)}{x}\right)^2 + (x+4)^2}$$

```
h = 5; l = 4;
x = linspace(1, 15);
L = sqrt((h*(x+l)./x).^2+(x+l).^2);
plot(x, L, ':b')
disp(min(L))  % 12.7017

syms x
f = sqrt((h*(x+l)./x).^2+(x+l).^2);
% fplot(f)
fplot(f, [2 10], 'r')
```

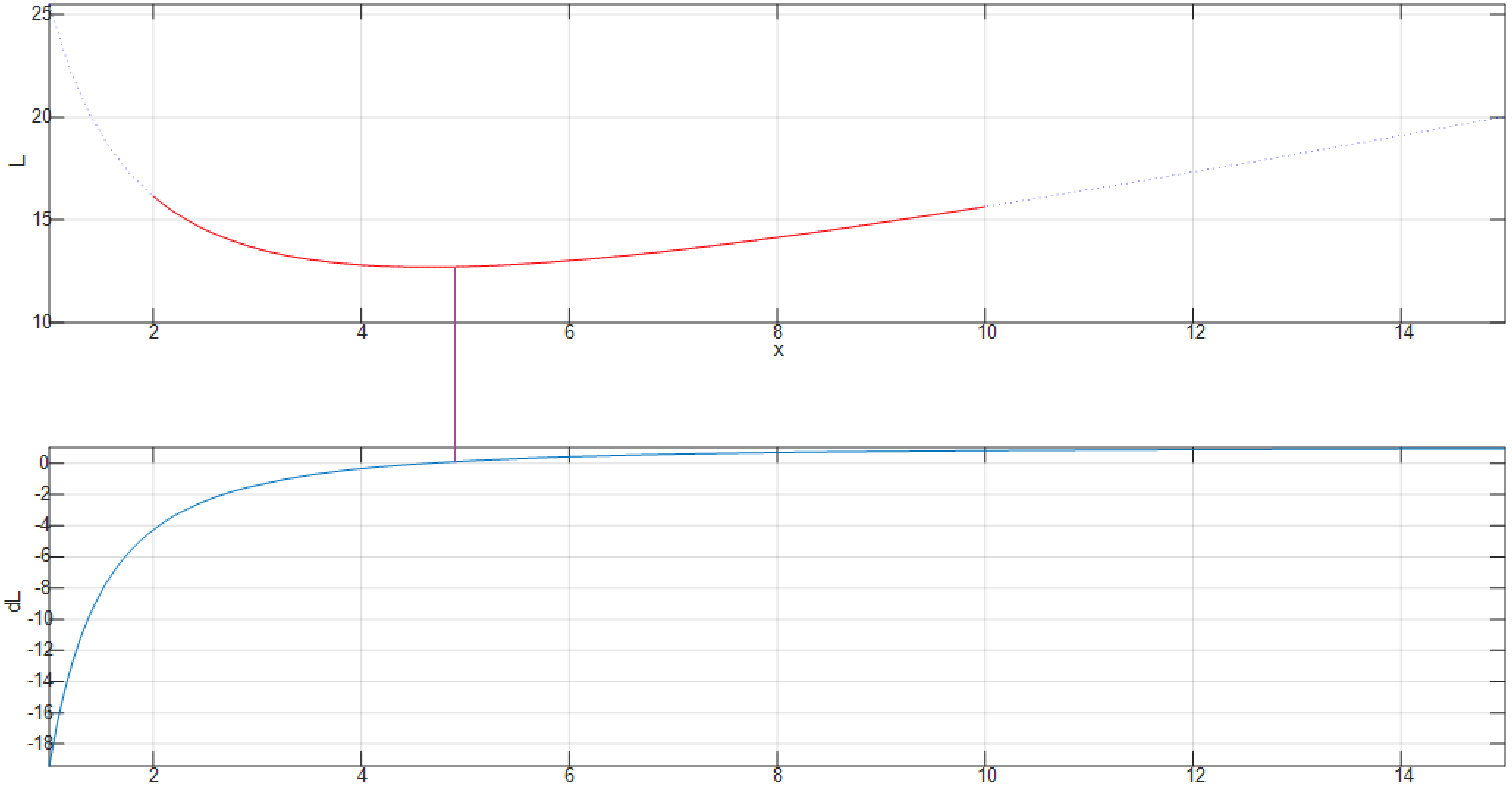


Example 4

```
syms x
f = sqrt((h*(x+l)./x).^2+(x+l).^2);
% fplot(f)
fplot(f, [2 10], 'r')

df = diff(f,x);
fplot(df, [1 15])
```




```
eval(solve(df, x))      % 4.6416, -2.3208 + 4.0197i, -2.3208 - 4.0197i
eval(subs(f, 4.6416))   % 12.7017
```



25
20
15
10
L
2
4
6
8
10
12
14
x
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
dL
2
4
6
8
10
12
14