An Alternative Approach for Making Maps Compatible with GPS[1]

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Abstract

The traditional geodetic networks were computed on a local reference ellipsoid. Nowadays, the artificial satellites determine the accurate three-dimensional coordinates in the World Geodetic System 1984 (WGS84). Thus, a point on the earth's surface could have two different coordinates. Therefore, it is necessary to relate the two by mathematical conversion in order to express coordinates in the same system. This paper describes an alternative approach for performing coordinate transformations between two different reference systems using a 3D conformal polynomial model which includes coefficients up to second order. In addition, a comparison between the proposed model and similarity transformation is carried out.

Keywords: conformal polynomial, seven parameter model, similarity transformation, transformation of 3D coordinates and datum shifts.

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1. Introduction

Ellipsoid that has been defined with orientation and position as well as size and shape is referred to as geodetic datum. Recently, satellite observations have been used to supply geodetic coordinates on a satellite or geocentric datum that have a physically meaningful and unambiguous definition of the origin and also have an immediate relation to the global systems such as those provided by Global Positioning System (GPS). However, terrestrial observations are able to assign geodetic coordinates on a local or non-geocentric datum.

The relationship between satellite and local geodetic datum is defined from the control points which their coordinates are usually calculated from terrestrial measurements of the geodetic network and compared to satellite datum coordinates. This comparison will result in defining the relation between the two, and the transformation parameters will be used to convert all new GPS derived coordinates into the local coordinate system. It should be noted that the good distribution of common points and their number play an important role on the accuracy of transformation model **[3]**.

If correct procedures are followed, the combined adjustment of two independent data sets will improve (1) the accuracy of the network by controlling the systematic errors and (2) the precision of the network because additional data are included. If systematic errors in scale and orientation exist within a geodetic network then not only will the coordinates be incorrect, but their estimated accuracy will be also optimistic **[5]**.

2. Bursa-Wolf Model

The Bursa-Wolf solver for a 7-parameter transformation is considered as the most common model in determining the datum transformation parameters between any different three-dimensional coordinate systems. It describes the relationship between the two coordinate systems by three shift components (δ_X, δ_Y, δ_Z), three rotation elements (R_X, R_Y, R_Z) and scale factor (K), figure (1). The rotations around the (X, Y, Z) axes are considered positive anti-clockwise when viewed from the positive end of the axis looking towards the origin.

The three-dimensional Cartesian coordinates of any station in the network defined in both systems under consideration (the common stations) are used as the observation of bursa-Wolf model. The mathematical equation of this model is given by **[10]**

$$\Re_i = (1 + K) \cdot R_T \cdot \Re_i + \Delta \qquad (1)$$

$$R_T = \begin{bmatrix} 1 & R_Z & -R_Y \\ -R_Z & 1 & R_X \\ R_Y & -R_X & 1 \end{bmatrix}$$

Where Δ is the translation vector between the two origins of the coordinate systems and R_T is the rotation matrix.

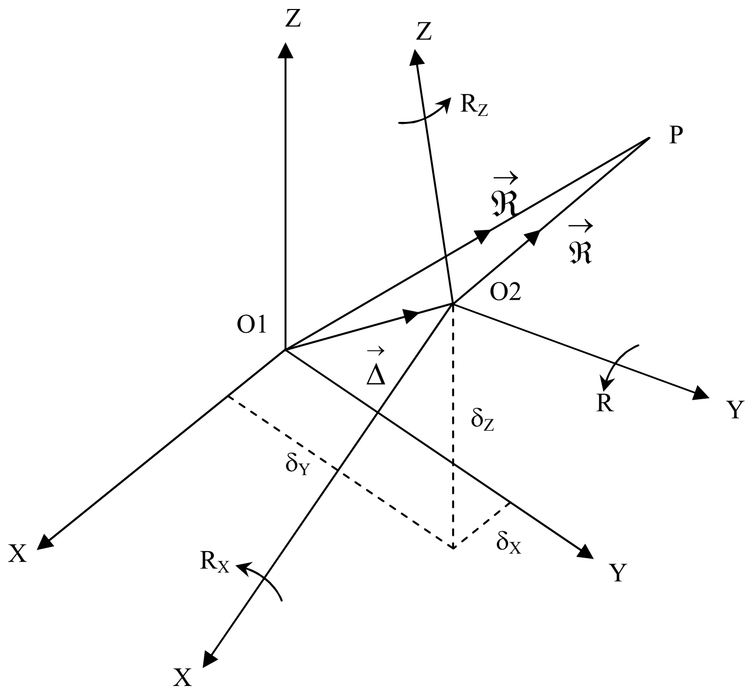

Fig. (1): The Bursa-Wolf Transformation

The relationships between Cartesian coordinates (X, Y, Z) and geodetic coordinates (φ, λ, h) of a point B related to an ellipsoid are shown in figure (2). The Z axis passes through the centre of the earth (or reference ellipsoid) and the poles, the X axis- passes through the centre and the Greenwich meridian, and the Y axis is at right angles to these. The Cartesian coordinates of a point B may be calculated by the following formula **[6]**:

$$X = (\nu + h)\cos\varphi\cos\lambda \qquad (2)$$

$$Y = (\nu + h)\cos\varphi\sin\lambda \qquad (3)$$

$$Z = (\nu(1 - e^2) + h)\sin\varphi \qquad (4)$$

The inverse computation of (φ, λ, h) from (X, Y, Z) can be made using the following **[2]**:

$$\lambda = \cos^{-1}\left(\frac{X}{P_f}\right) \qquad (5)$$

$$\varphi = \tan^{-1}\left(\frac{Z + e'^2 \cdot b \cdot \sin^3\omega}{P_f - e^2 \cdot a \cdot \cos^3\omega}\right) \qquad (6)$$

$$h = \frac{P_f}{\cos\varphi} - \nu \qquad (6)$$

Where

$$\omega = \tan^{-1}\left(\frac{Z \cdot a}{P_f \cdot b}\right)$$

$$P_f = \sqrt{(X^2 + Y^2)}$$

e first eccentricity of an ellipsoid.
a, b semi major-axis and semi minor-axis of an ellipsoid.
ν radius of curvature in the prime vertical plane.

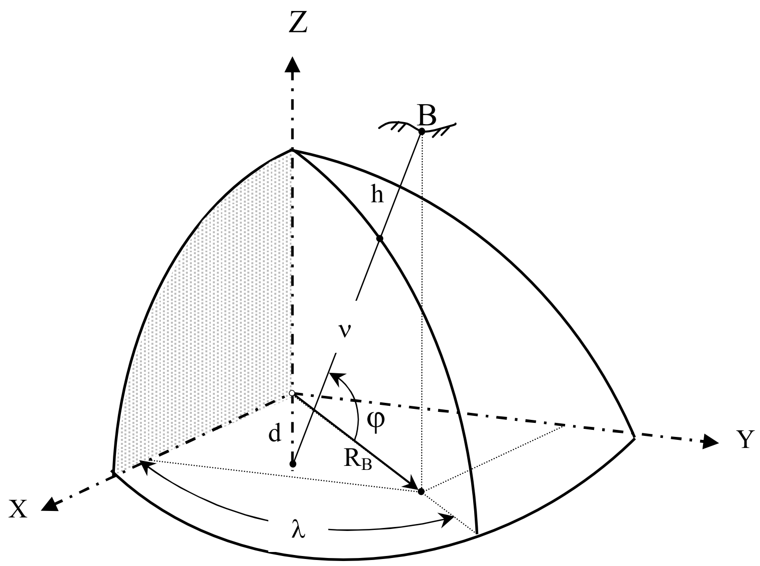

Fig. (2): Relation between Cartesian and Geodetic Coordinates

3. The Developed Approach

The multiple regression equations were used by the Defense Mapping Agency **[9]** to transform geodetic coordinates (latitude and longitude) between two datums. Also, **[1]** used this approach to express the variable datum shifts relative to geodetic coordinates. The multiple regression equations depend on very complex polynomials, where every equation has coefficients not correlated with another. In addition the conversion of these equations is not conformal. Nevertheless, the alternative approach to transform a 3D GPS coordinates into the terrestrial networks will be discussed here based on a conformal polynomial procedure.

The fundamental equations of the developed mathematical model are:

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix}_{i(Local)} = \begin{vmatrix} x \\ y \\ z \end{vmatrix}_{i(WGS84)} + \begin{vmatrix} X_o \\ Y_o \\ Z_o \end{vmatrix}_{i} \quad (8)$$

In which (x, y, z) and (X, Y, Z) are the Cartesian coordinates in WGS84 datum and local geodetic datum, respectively, and (X_o, Y_o, Z_o) are variable datum shifts at the station i. This can be presented in vector notions as follows:

$$\Re_i = \Re_i + \Delta \qquad (9)$$

In which $\Re$ and $\Re$ are position vectors and Δ is a vector of datum shifts. The approach is based on finding the variable datum shift components using a 3D conformal polynomial model includes coefficients up to second order.

The general formula of three-dimensional transformation using polynomials is **[7]**

$$X_o = a_o + a_1\varphi + a_2\lambda + a_3 h + a_4\varphi^2 + a_5\lambda^2 + a_6 h^2 + a_7\varphi\lambda + a_8\lambda h + a_9\varphi h \qquad (10)$$

$$Y_o = b_o + b_1\varphi + b_2\lambda + b_3 h + b_4\varphi^2 + b_5\lambda^2 + b_6 h^2 + b_7\varphi\lambda + b_8\lambda h + b_9\varphi h \qquad (11)$$

$$Z_o = c_o + c_1\varphi + c_2\lambda + c_3 h + c_4\varphi^2 + c_5\lambda^2 + c_6 h^2 + c_7\varphi\lambda + c_8\lambda h + c_9\varphi h \qquad (12)$$

The conformal conversion does not exist in three dimensions beyond the first order (or linear) case given by seven parameter transformation. A close approximation, which is available for only second-degree terms, is derived by imposing the conditions of Cauchy- Riemann equation on every pair of coordinates in equation (10), (11) and (12). This makes the projections of three-dimensional space onto each of three planes conformal.

Applying the following on the general equations of polynomials**:**

$$\frac{\partial X_o}{\partial \varphi} = \frac{\partial Y_o}{\partial \lambda} = \frac{\partial Z_o}{\partial h}$$
$$\frac{\partial X_o}{\partial \lambda} = -\frac{\partial Y_o}{\partial \varphi}, \quad \frac{\partial X_o}{\partial h} = -\frac{\partial Z_o}{\partial \varphi}, \quad \frac{\partial Y_o}{\partial h} = -\frac{\partial Z_o}{\partial \lambda} \qquad (13)$$

Leads to

$$X_o = A_o + A\varphi + B\lambda - Ch + E\left(\varphi^2 - \lambda^2 - h^2\right) + 0 + 2F\varphi\lambda + 2G\varphi h \qquad (14)$$

$$Y_o = B_o - B\varphi + A\lambda + Dh - F\left(\varphi^2 - \lambda^2 + h^2\right) + 2E\varphi\lambda + 0 + 2G\lambda h \quad (15)$$

$$Z_o = C_o + C\varphi - D\lambda + Ah - G\left(\varphi^2 + \lambda^2 - h^2\right) + 2F\lambda h + 2E\varphi h + 0 \quad (16)$$

The polynomial coefficients are assigned from the data of those points, which are known coordinates in both systems, i.e. common stations and using least squares adjustment. These transformation parameters are then applied to convert the GPS coordinates of the other points into corresponding ground coordinates.

The least squares adjustment is based on the principle of minimizing the sum of squares of the observation corrections, known as residuals. There are two techniques of this approach are most commonly employed, namely the parametric and conditional techniques. Generally, the geodesists prefer to use the parametric technique for adjustment problem because it can easily be programmed to take directly the observations with its measured precision and get all quantities required from adjustment process. The mathematical model of the parametric technique is observation equations of all measurements that must be in a linear form. This form usually is written in matrix notation as **[7]**:

$$V + B \cdot \Delta = f \quad (17)$$

Where

V vector of observational residuals.

B numerical coefficients matrix of parameters.

Δ vector of unknown parameters or corrections.

f column vector of numerical constants.

First, approximate values of the polynomial coefficients are determined. Hence, ℓ_i, $\ell_i^{/}$ and $\ell_i^{//}$ components are calculated from (8) as follows

$$\ell_i = (X - X)_i - \begin{Bmatrix} A_o + A\varphi_i + B\lambda_i - Ch + E(\varphi_i^2 - \lambda_i^2 - h_i^2) + \\ 2F\varphi_i\lambda_i + 2G\varphi_i h_i \end{Bmatrix} \quad (18)$$

$$\ell_i^{/} = (Y - Y)_i - \begin{Bmatrix} B_o - B\varphi_i + A\lambda_i + Dh - F(\varphi_i^2 - \lambda_i^2 + h_i^2) + \\ 2E\varphi_i\lambda_i + 2G\lambda_i h_i \end{Bmatrix} \quad (19)$$

$$\ell_i^{//} = (Z - Z)_i - \begin{Bmatrix} C_o - C\varphi_i + D\lambda_i + Ah - G(\varphi_i^2 + \lambda_i^2 - h_i^2) + \\ 2Fh_i\lambda_i + 2E\varphi_i h_i \end{Bmatrix} \quad (20)$$

Then, the transpose vector of numerical constants and numerical coefficients matrix of parameters is:

$$f^t = \begin{bmatrix} \ell_1 & \ell_1^{/} & \ell_1^{//} & \cdots\cdots & \ell_n & \ell_n^{/} & \ell_n^{//} \end{bmatrix} \qquad (21)$$

The corrections transpose vector of the polynomial coefficients and observational residuals have the following forms:

$$\Delta^t = \begin{bmatrix} dA_{o_o} & dA_o & dB_o & dC_o & dD_o & dE_o & dF_o & dG_o & dB_{o_o} & dC_{o_o} \end{bmatrix} \qquad (22)$$

$$V^t = \begin{bmatrix} V_{X_1} & V_{Y_1} & V_{Z_1} & \cdots\cdots & V_{X_n} & V_{Y_n} & V_{Z_n} \end{bmatrix} \qquad (23)$$

Numerical coefficients matrix of parameters is:

$$B = \begin{bmatrix} 1 & \varphi_1 & \lambda_1 & -h_1 & 0 & (\varphi_1^2 - \lambda_1^2 - h_1^2) & 2\,\varphi_1\,\lambda_1 & 2\,\varphi_1\,h_1 & 0 & 0 \\ 0 & \lambda_1 & -\varphi_1 & 0 & h_1 & 2\,\varphi_1\,\lambda_1 & -(\varphi_1^2 - \lambda_1^2 + h_1^2) & 2\,\varphi_1\,h_1 & 1 & 0 \\ 0 & h_1 & 0 & \varphi_1 & -\lambda_1 & 2\,\varphi_1\,h_1 & 2\,\lambda_1\,h_1 & -(\varphi_1^2 + \lambda_1^2 - h_1^2) & 0 & 1 \\ \text{-----} & & & & & & & & & \\ 1 & \varphi_n & \lambda_n & -h_n & 0 & (\varphi_n^2 - \lambda_n^2 - h_n^2) & 2\,\varphi_n\,\lambda_n & 2\,\varphi_n\,h_n & 0 & 0 \\ 0 & \lambda_n & -\varphi_n & 0 & h_n & 2\,\varphi_n\,\lambda_n & -(\varphi_n^2 - \lambda_n^2 + h_n^2) & 2\,\varphi_n\,h_n & 1 & 0 \\ 0 & h_n & 0 & \varphi_n & -\lambda_n & 2\,\varphi_n\,h_n & 2\,\lambda_n\,h_n & -(\varphi_n^2 + \lambda_n^2 - h_n^2) & 0 & 1 \end{bmatrix} \qquad (24)$$

4. Applied Study And Results

The available data of the application are demonstrated in figure (3), which are the coordinates of only 16 first order Australian network stations in the Australian Geodetic Datum (AGD) and in WGS84 **[3]**. The results accuracy is depending on the measurements accuracy in both systems and the suitable coverage of data in the area under study, therefore, that any errors in the employed data will affect the results negatively. This means that it is better to use the first order stations only in good coverage and adequate spacing to get accurate results and well consistent coordinates from the conversion processing.

For comparison purposes, the Bursa-Wolf conversion model has been taken as an example of the classical method to be applied against the developed technique. In addition, a computer program is written by the author for these models using Visual C++ language, figure (4). This software has the possibility of entering the geodetic coordinates of common stations and the output data of it are:

- The transformation parameters with their standard deviations.
- The observation residuals, the covariance matrices of both the coordinates and the unknown transformation parameters.

- The posteriori variance factor.
- The three–dimensional geodetic coordinates of the new points that are transformed to the local system according to the satellite coordinates.

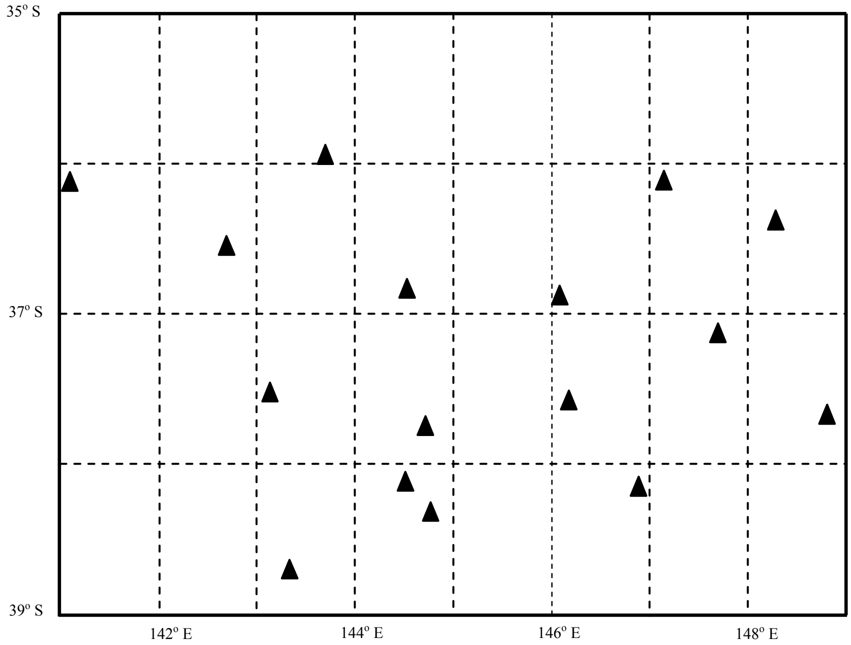

Fig. (3): Distribution of Common Stations

The obtained results from the software for the two transformation models are listed in table (1) to (4). Table (1) and (3) show the transformation parameters and their standard deviations (σ_Δ) while table (2) and (4) give the minimum, maximum, average of the residuals, standard deviations of single value determination (σ_r) and unit weight (σ_o).

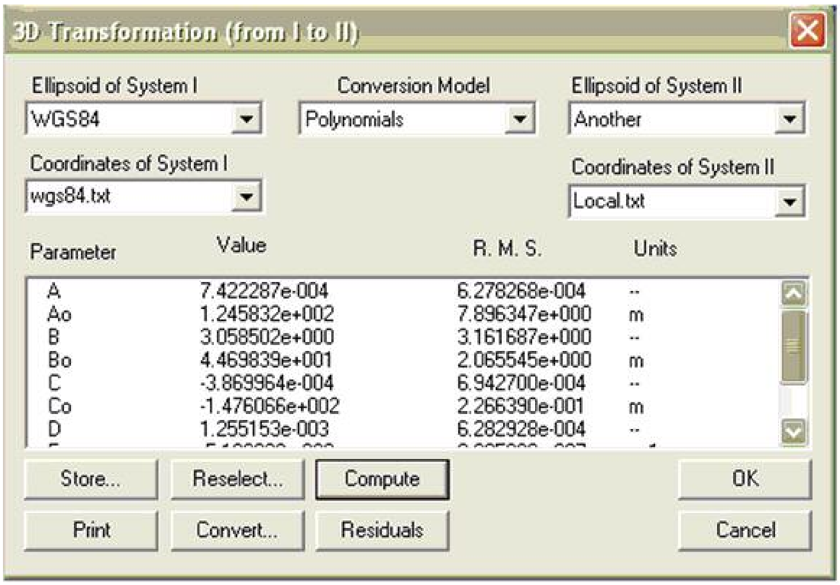

Fig. (4): Dialog Box of the 3D Transformation Models

Table (1): The Bursa-Wolf Transformation Model

Parameter	Estimated Value	σ_Δ	Units
δ_X	129.736	± 2.580	m
δ_Y	57.419	± 2.123	m
δ_Z	-166.007	± 3.326	m
R_X	0.160	± 0.085	sec
R_Y	-0.507	± 0.112	sec
R_Z	0.042	± 0.059	sec
K	-2.80	± 0.268	ppm

Table (2): Statistics of the Residuals (in Meters) at Common

Stations from Bursa-Wolf Model

Component	V_X	V_Y	V_Z
Minimum	-0.018	-0.002	-0.009
Maximum	0.474	0.509	0.526
Average	0.000	0.000	0.000
σ_r	± 0.241	± 0.209	± 0.203
σ_o	± 0.229		

Table (3): Polynomial Coefficients

Parameter	Estimated Value	σ_Δ	Units
A_o	124.583	± 7.896	m
B_o	44.984	± 2.066	m
C_o	-147.607	± 22.664	m
A	$7.422*10^{-4}$	$\pm 6.278*10^{-4}$	-
B	3.058	± 3.162	-
C	$3.058*10^{-4}$	$\pm 6.943*10^{-4}$	-
D	$1.255*10^{-3}$	$\pm 6.283*10^{-4}$	-
E	$-5.169*10^{-8}$	$\pm 2.896*10^{-7}$	m^{-1}
F	$3.998*10^{-7}$	$\pm 2.791*10^{-7}$	m^{-1}
G	$-8.757*10^{-8}$	$\pm 2.784*10^{-7}$	m^{-1}

Table (4): Statistics of the Residuals (in Meters) at Common

Stations from Developed Approach

Component	V_X	V_Y	V_Z
Minimum	**-0.049**	**-0.123**	**-0.165**
Maximum	**0.569**	**1.126**	**0.680**
Average	**0.000**	**0.038**	**0.000**
σ_r	**± 0.338**	**± 0.470**	**± 0.312**
σ_o	**± 0.414**		

5. Analysis Of Transformation Results

Firstly, from table (2) and (4), the standard deviations of the residuals are:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \sigma_o\end{bmatrix} = \begin{bmatrix}\pm 0.241\\ \pm 0.209\\ \pm 0.203\\ \pm 0.229\end{bmatrix}_{7-Par.} , \begin{bmatrix}\pm 0.338\\ \pm 0.470\\ \pm 0.312\\ \pm 0.414\end{bmatrix}_{Poly.} \quad m$$

These standard deviations reflect the accuracy of the model fitting to the common stations. The fitting accuracy of polynomial approach is closed with insignificant difference to 7-parameter model. However, more data points give the best polynomial coefficients for the datum shift components as they make the selection process of the good points, which achieve a preferable solution, are more realistic.

This case of study can not show the conditions of proper distribution of common stations because the number of these points was not sufficient. More investigation can be done on a good sample with enough common points to satisfy the required strong distribution.

The comparison between approaches is concerted on the residuals and accuracy of the geodetic coordinates of 16 data points (check stations) that reproduced from the derived transformation parameters. The statistics of the differences between the actual and computed coordinates using the two conversions models are summarized in table (5) and (6).

Table (5): Statistics of the Differences and their Vectors at Check Stations between the Data and the reproduce AGD Geodetic Coordinates Using Bursa-Wolf Transformation Model

Component	φ	λ	h	Vector
Units	sec	sec	m	m
Minimum	-0.004	-0.012	-0.146	0.471
Maximum	0.056	0.056	0.493	2.482
Average	0.021	-0.001	-0.196	1.352
σ_r	0.025	0.033	0.335	0.523

Table (6): Statistics of the Differences and their Vectors at Check Stations between the Data and the Reproduce AGD Geodetic Coordinates Using Polynomial Model

Component	φ	λ	h	Vector
Units	sec	sec	m	m
Minimum	-0.008	-0.015	-0.003	0.120
Maximum	0.014	0.030	0.293	0.981
Average	-0.006	0.013	0.094	0.642
σ_r	0.005	0.016	0.179	0.301

The means and standard deviations of the differences and their vectors in the preceding tables are written below:

$$\begin{bmatrix} \bar{\varphi} \pm \sigma_\varphi \\ \bar{\lambda} \pm \sigma_\lambda \\ \bar{h} \pm \sigma_h \\ \bar{\vartheta} \pm \sigma_\vartheta \end{bmatrix} = \begin{bmatrix} 0.021 \pm 0.025 \\ -0.001 \pm 0.033 \\ -0.196 \pm 0.335 \\ 1.352 \pm 0.523 \end{bmatrix}_{7-Par.} , \begin{bmatrix} 0.006 \pm 0.005 \\ 0.013 \pm 0.016 \\ 0.094 \pm 0.179 \\ 0.642 \pm 0.301 \end{bmatrix}_{Poly.} \quad m$$

These values indicate that developed model is able to predict coordinates and heights to accuracy better than 7-parameter transformation. The

application shows a great improvement in the ellipsoidal height for polynomial case, this covers the defect of using GPS, where normally the problem in height determination. Also, the residuals of latitude and vector are much better in polynomial than utilizing 7-parameter model.
However, the vector values that resulted from the given check point coordinates and those computed using Bursa-Wolf and polynomial transformations are shown in figure (5). The discrepancies in the vectors from Bursa-Wolf model are larger than in the case of the polynomial transformation.

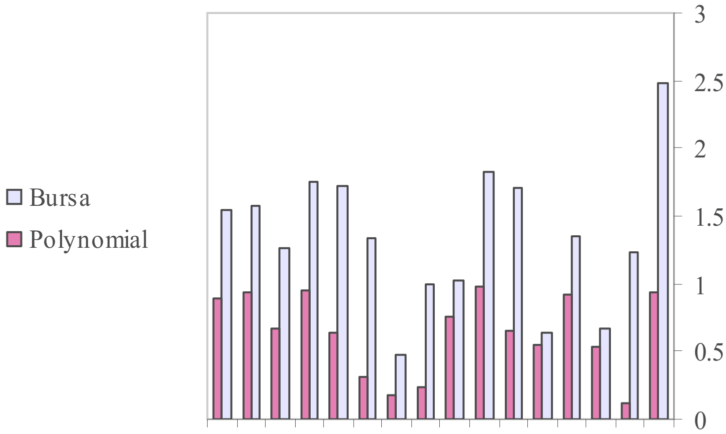

Fig. (5): the Values of Vector (in Meters) at Check Stations from Bursa-Wolf and Polynomial Transformations

6. Conclusions

A position of a point on the earth's surface can be defined in terms of its latitude, longitude and height. The expression of such a position requires a reference ellipsoid. The defining parameters of the reference ellipsoid, its position and orientation, are known as the geodetic datum. The traditional geodetic datum differs from that used by the GPS. Although a transformation model between these datums is available, but in this work an alternative approach for converting from one datum to another, often called a datum shift, is developed.
Coordinates conversion is a process of determining and applying the relationship between two sets of coordinates. According to the transformation result obtained in this study, the following conclusions can be found:

- The developed mathematical model is based on a 3D conversion employing 2nd order polynomial that fulfills the conformality condition and requires the geoid heights in the old datum at the common points. This new method has been tested with data from the Australian network and compared with Bursa 7-parameter similarity transformation to confirm its validity.
- The differences vector in the case of the developed approach is lesser than the seven parameter transformation. This fact reflects the accuracy of the proposed mathematical model that may be attributed to the polynomial degree.
- Assigning of polynomial coefficients using least squares solution has two advantages: (1) it distributes the errors of fit so that the maximum residual error is minimized and (2) it furnishes residuals for all data points and the standard error of single observation, thus making it unnecessary to perform any additional accuracy tests.
- In least squares solution, a posterior precision estimation of parameters and residuals are possible and can provide valuable information when choosing one transformation method in preference to others.
- This model is proved to be simple to understand, easy to apply and gives sufficiently accurate results for most purposes. It is verified that it is possible to use it as an alternative to 7-parameter transformation based on datum shifts.
- Further investigation is necessary to draw conclusions when a large number of common stations are available.

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