

The Calculation of Structure Characteristics From a Given Model or Measured Time Responses Using Orthogonal Transformations

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Abstract

Two orthogonal Transformation techniques; namely Householder's, and QR methods, are adopted together as a procedure to find a flexible structure natural frequencies, and their corresponding mode shapes from a proposed model or time responses data. The approach shows its quiet good ability in calculating these parameters from simulated systems. Moreover, the method has the capability to identify the natural frequencies, from experimental data that have finite frequency range, with good accuracy.

1- Introduction

It is important to model a simple structure by lumped masses that are connected together, and to provide supports by springs of known

stiffnesses. There are many approaches, which are developed to calculate the system natural frequencies and their corresponding mode shapes, from the proposed models [1] and [2]. Moreover, the time responses at the lumped masses can be determined by well-known analytical and numerical procedures. For more details see [3], and [4]. The validity of this suggested model can be checked by comparing the calculated output responses of the assumed model with the measured ones [5] and [6].

Furthermore, It can be accepted if the system parameters of the given model are the same as those found from measured time or frequency data responses [5] [7], and [8]. On the other hand, it was found that the structure characteristics could be identified from measured frequency response data using principal component analysis (PCA) [5], and [9].

In the present study, orthogonal transformation routines are going to be implemented to calculate or estimate the structure parameters from a given model or measured time output displacement responses. A software program is developed to implement these transformations on theoretical and experimental data. The presented approach is applied on simulated examples to demonstrate its capability to find the system parameters. In addition, the approach has the ability to estimate the natural frequencies in the frequency range of interest, which are less than the number of frequencies for the assumed model, with good accuracy.

2. Theoretical Background

A vibrating structure may be modeled by N finite number of lumped masses, which are connected to each other and to supports via springs as shown in Figure 1. The dynamic behavior of the structure can be found using Newton's second law in matrix form as,

$$M\ddot{x} + Kx = f \quad (2.1)$$

where,

M is the diagonal mass matrix of dimension (N x N)

K is the symmetric stiffness matrix of dimension (N x N)

x, \ddot{x} are the displacement, and the acceleration response vectors of dimension (N x 1)

f is the applied force vector of dimension (N x 1)

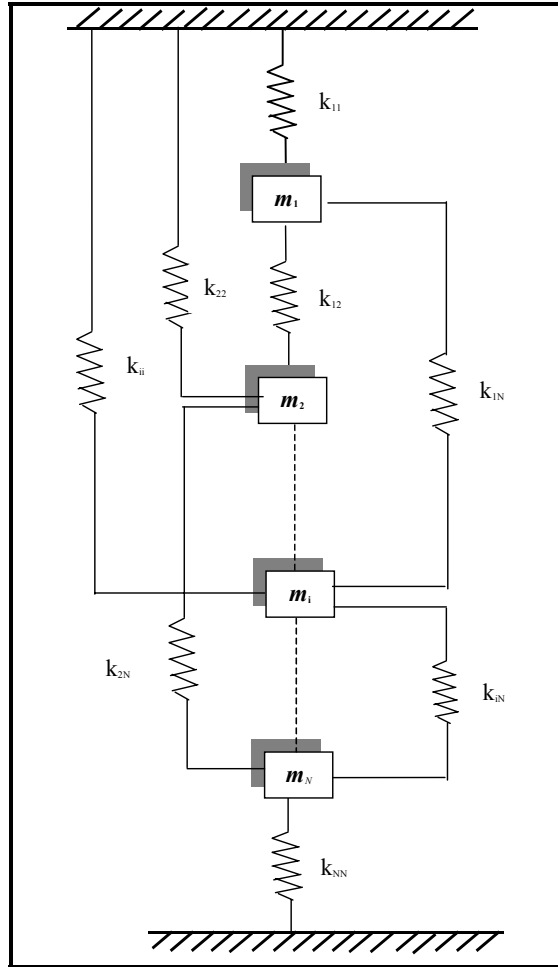


Figure 1: The lumped model for any structure under consideration

The homogeneous solution of equation (2.1) can be determined by letting the vector f equal zero. Thus,

$$M\ddot{x} + Kx = 0 \quad (2.2)$$

Now, let the displacement response vector x be represented by the harmonic form of,

$$x = X e^{i\omega t} \quad (2.3)$$

$$\therefore \ddot{x} = -\omega^2 X e^{i\omega t} \quad (2.4)$$

where X is the amplitude vector, and ω is the frequency. Substitute equations (2.3), and (2.4) into equation (2.2) yield,

$$(K - \omega^2 M)X = 0 \quad (2.5)$$

Dividing both sides of equation (2.5) by M^{-1} results in,

$$(A - \lambda I) = 0.0 \quad (2.6)$$

where, $A = M^{-1}K$, $\lambda = \omega^2$, and I is an identity matrix. The solution of equation (2.6) by different procedures, such as iterative and power methods, provides two of the most important parameters of any vibrating structure [3]. These are the diagonal undamped natural frequencies matrix Ω , and their corresponding normalized mode shapes matrix Ψ .

However, one can reach this goal by factorizing the matrix A to three matrices using orthogonal transformations as follows [10],

$$A = U \Sigma U^T \quad (2.7)$$

where,

U is a matrix with an orthonormal vectors, which will represent the mass normalized mode shapes matrix Φ .

Σ is a diagonal matrix with the natural frequencies are its diagonal elements

It is easy to construct from the U matrix the normalized mode shapes matrix Ψ by dividing each column by its first elements. These orthogonal transformations are the Householder and QR methods, which are going to be discussed in the next two sections.

2.1 Householder's Method (HM)

The Householder's method is used to transform a symmetric matrix A of dimension $(N \times N)$, to a symmetric tridiagonal matrix B . The procedure starts by building an orthogonal matrix $H^{(1)}$ which has the form of [11],

$$H^{(1)} = I - 2hh^T \tag{2.8}$$

where, I is an identity matrix, and h is a vector of dimensions (N×N), and (N×1) respectively. The vector h is chosen to be orthonormal vector, so that the matrix H⁽¹⁾ will be orthonormal, where both of them satisfy the property of,

$$h^T h = 1 \tag{2.9}$$

$$H^{(1)T} H^{(1)} = I \tag{2.1.0}$$

The matrix H⁽¹⁾ will be pre- and post- multiplied by the matrix A to produce a new matrix A⁽²⁾ as,

$$A^{(2)} = H^{(1)} A H^{(1)} \tag{2.11}$$

with the following characteristics,

$$a_{11}^{(2)} = a_{11} \tag{2.12}$$

$$a_{21}^{(2)} = a_{12}^{(2)} = \alpha \tag{2.13}$$

$$a_{j1}^{(2)} = a_{1j}^{(2)} = 0.0 \text{ for } j = 3,4,\dots, N \tag{2.14}$$

where, α is a constant, while a_{j1}⁽²⁾, and a_{1j}⁽²⁾ are the jth elements of the first column, and the first row of matrix A⁽²⁾ respectively. To preserve the first property, the first element h₁ of the vector h should be zero. Our goal now is to find the elements h₂, h₃, ..., h_N of h so that the other properties of A⁽²⁾ are fulfilled. Thus, let us construct the following sub-matrix equation,

$$\hat{y}^{(2)} = \hat{H}^{(1)} \hat{y} \tag{2.15}$$

where:

$$\hat{y}^{(2)} = (\alpha, 0, \dots, 0)^T \tag{2.16}$$

$$\hat{H}^{(1)} = I_{N-1} - 2\hat{h}\hat{h}^T \tag{2.17}$$

$$\hat{h} = (h_2, h_3, \dots, h_N)^T \tag{2.18}$$

$$\hat{y} = (a_{21}, a_{31}, \dots, a_{N1})^T \tag{2.19}$$

Equation (2.15) can be written as,

$$\hat{y}^{(2)} = \hat{y} - 2\hat{h}\hat{h}^T\hat{y} \quad (2.20)$$

As the multiplication of $\hat{h}^T\hat{y}$ yields a single value, which is assumed to be r . Therefore, equation (2.21) becomes,

$$\hat{y}^{(2)} = \hat{y} - 2r\hat{h} \quad (2.21)$$

or,

$$(\alpha, 0, \dots, 0)^T = (a_{21} - 2rh_2, a_{31} - 2rh_3, \dots, a_{N1} - 2rh_N)^T \quad (2.22)$$

Equating the elements on both sides of equation (2.22) will produce (N-1) algebraic equations. These are,

$$h_2 = \frac{a_{21} - \alpha}{2r} \quad (2.23)$$

$$h_j = \frac{a_{j1}}{2r} \quad j = 3, 4, \dots, N \quad (2.24)$$

Then, if these (N-1) equations are squared, and the results are added, a new algebraic equation will be formulated,

$$4r^2 \sum_{j=2}^N h_j^2 = \sum_{j=2}^N a_{j1}^2 - 2\alpha a_{21} + \alpha^2 \quad (2.25)$$

Since, equations (2.9), and $\hat{y}^T y$ can provide us by,

$$\sum_{j=2}^N h_j^2 = 1 \quad (2.26)$$

$$\alpha^2 = \sum_{j=2}^N a_{j1}^2 \quad (2.27)$$

From equations (2.23) and (2.24), it is clear that to ensure this routine is working, r should not be equal to zero. Thus, to prevent halting of this process, it was found that α should be selected to have the value of [11],

$$\alpha = -(\text{sgn } a_{21}) \left(\sum_{j=2}^N a_{j1}^2 \right)^{1/2} \quad (2.28)$$

Consequently,

$$r = \left(\frac{1}{2}\alpha^2 - \frac{1}{2}a_{21}\alpha\right)^{1/2} \tag{2.29}$$

As a result, the elements of the vector h can be determined from equations (2.24), and (2.25). Then $H^{(1)}$ is calculated utilizing equation (2.8), and, finally the matrix $A^{(2)}$ is obtained from equation (2.11), which has the form of,

$$A^{(2)} = H^{(1)}AH^{(1)} = \begin{bmatrix} a_{11} & \alpha & 0 & 0 & 0 & 0 & 0 \\ \alpha & a_{22}^{(2)} & . & . & . & . & a_{2N}^{(2)} \\ 0 & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . \\ 0 & a_{N2}^{(2)} & . & . & . & . & a_{NN}^{(2)} \end{bmatrix} \tag{2.30}$$

Repeating the procedure for $(N-2)$ times to form the tridiagonal symmetric matrix B , i.e.,

$$\therefore B = H^{(N-2)}H^{(N-3)} \dots H^{(1)}AH^{(1)} \dots H^{(N-3)}H^{(N-2)} \tag{2.31}$$

2.2 QR Method

The QR approach is based on transforming the resulting tridiagonal matrix B to an upper tridiagonal matrix R by multiplying it with an orthogonal matrix Q . This can be expressed as,

$$R^{(1)} = Q^{(1)}B \tag{2.32}$$

The $Q^{(1)}$ matrix is constructed from the multiplication of $(N-1)$ orthogonal matrices which are denoted by the symbol P . Each one of these orthogonal matrices is generated to make one of the lower off diagonal elements of the matrix B equal zero,

$$\therefore Q^{(1)} = P_N P_{N-1} \dots P_2 \tag{2.33}$$

where, the matrix P_{k+1} has the form of,

$$P_{k+1} = \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & c_{k+1} & s_{k+1} & 0 \\ 0 & -s_{k+1} & c_{k+1} & 0 \\ 0 & 0 & 0 & I_{n-k-1} \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{Row } k \\ \\ \uparrow \text{Column } k \end{array} \quad k=1,2,\dots,N \quad (2.34)$$

The constants c_k , and s_k in matrix P_{k+1} are determined using the following formulae,

$$s_{k+1} = \frac{b_{k+1 \ k}}{\sqrt{b_{k+1 \ k}^2 + b_{kk}^2}} \quad (2.35)$$

$$c_{k+1} = \frac{b_{kk}}{\sqrt{b_{k+1 \ k}^2 + b_{kk}^2}} \quad (2.36)$$

where, the constants b 's, in equations (2.35) and (2.36), are the elements of the matrix B . After constructing the $Q^{(1)}$ matrix and calculating the $R^{(1)}$ matrix, a new tridiagonal matrix $B^{(1)}$ can be formed from these two matrices, i.e.,

$$B^{(1)} = R^{(1)}Q^{(1)} \quad (2.37)$$

The values of the lower, and the upper off-diagonal elements of matrix $B^{(1)}$ are smaller than those of matrix B . Thus, the preceding process can be repeated until the values of all the off-diagonal elements are diminished. The final new B matrix represents the diagonal Σ matrix, while the multiplication of all the Q , and H matrices will produce the orthonormal matrix U . It is worth mentioning that, the processing time can be reduced significantly, if the shifting routine is added to it [12].

2.3 Principal Component Analysis (PCA)

PCA is a statistical procedure, which was used in structural vibration identifications from measured frequency response data [13], [5], and [14].

PCA main task is to transform coupled output responses of vibrating structure to new de-coupled responses. The approach begins by preparing an equivalent covariance matrix S from the measured responses utilizing the following formula [5],

$$S = X^T X \quad (2.38)$$

where,

X is the time responses matrix measured at N locations on the structure with N_p sampled data each, the matrix dimension is

$$N_p \times N$$

S is the covariance matrix of dimension $(N_o \times N)$

Then, orthogonal transformation techniques are adopted to factorize the S matrix to,

$$S = U^T \Sigma U \quad (2.39)$$

where,

U is a matrix with orthonormal columns of dimension $(N \times N)$

Σ is a diagonal matrix with elements of descending order, of dimension $(N \times N)$

Since, the measured data usually include N_d of natural frequencies where $N_d \leq N$, these natural frequencies are well separated ones in simple structures. It was found that the value of the elements in the diagonal matrix Σ are of descending order, with only the first N_d elements have significant values, [5], and [9]. As a result, the covariance matrix S can be constructed without losing valuable information using the following expression,

$$S \cong U'^T \Sigma' U' \quad (2.40)$$

where, U' is an orthonormal matrix of dimension $(N_d \times N)$, and Σ' is a

diagonal matrix of dimension $(N_d \times N_d)$. In fact, the columns of matrix

U' are the mass normalized mode shapes of the system which are included in the recorded data. Thus, they can be utilized to transform the N coupled responses in the X matrix to the new N_d de-coupled

responses, saved in the X' matrix, as follows,

$$X' = U'^T X \quad (2.41)$$

Finally, the new de-coupled responses represent single degrees of freedom systems. Consequently, it becomes easy and efficient to estimate the system natural frequencies from them.

3. Simulation Results

The approach is going to be implemented on a number of systems to demonstrate its capability of calculating as well as estimating the excited natural frequencies and their corresponding mode shapes.

These systems are:

3.1 Three Degrees of Freedom System

An ideal system is constructed with a diagonal mass matrix M where its three elements, m_{11} , m_{22} , and m_{33} are equal, and have the magnitude of 1kg. The stiffness matrix is chosen to be,

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \times 10^4 \frac{N}{m}$$

The matrix A is calculated, then the orthogonal transformation techniques are utilized to factorize A to three matrices as given in equation (2.7). The

calculated Ω , and Φ matrices represent the system three natural frequencies and their corresponding mass normalized mode shapes matrices respectively. These are,

$$\Omega = \begin{bmatrix} 76.53 & 0 & 0 \\ 0 & 141.43 & 0 \\ 0 & 0 & 184.79 \end{bmatrix} \text{ rad/sec} \quad \Phi = \begin{bmatrix} 0.50 & 0.707 & 0.50 \\ 0.707 & 0.00 & -0.707 \\ 0.50 & -0.707 & 0.50 \end{bmatrix}$$

The normalized mode shapes matrix can be determined by dividing each column in the matrix Φ by its first element,

$$\therefore \Psi = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 1.41 & 0 & -1.41 \\ 1.0 & -1.0 & 1.0 \end{bmatrix}$$

Now, a force of magnitude 0.1 N is applied at mass 1 and the output displacements at the three masses are calculated using the mode superposition method, Craig [3]. A 1024 sampled data are recorded for each output, and the sampling time dt used here is 0.01 sec. Since the cutoff frequency of the FFT is 50 Hz, all the three system natural frequencies of the system are excited. The calculated displacement outputs of the system are given in Figure 2, and can be assumed as measured output by displacement sensors. It is clear that it is difficult to estimate the system natural frequencies, and their corresponding mode shapes from these time outputs. Thus, the orthogonal transformation approach is adopted to generate a new de-coupled output from the system original coupled outputs. To achieve this goal, the matrix S is obtained from the system original coupled outputs, then the orthogonal transformation technique is applied to factorize it, and get the orthonormal matrix U which was found to be,

$$U = \begin{bmatrix} 0.50 & 0.7073 & 0.4997 \\ 0.7069 & -9.931 \times 10^{-5} & -0.7073 \\ 0.5002 & -0.7069 & 0.5001 \end{bmatrix}$$

It is obvious that the U matrix is actually the Φ matrix of the system, and the Ψ matrix is then calculated from U as explained previously. The de-coupled single DOF outputs are then determined utilizing equation (2.41), and the results are shown in Figure 3. Now, the system natural frequencies can be estimated directly from either the time or frequency of those outputs. The values of these frequencies are,

$$\omega_{n1} = 76.4 \text{ rad/sec} \quad \omega_{n2} = 142.91 \text{ rad/sec} \quad \omega_{n3} = 184.8 \text{ rad/sec}$$

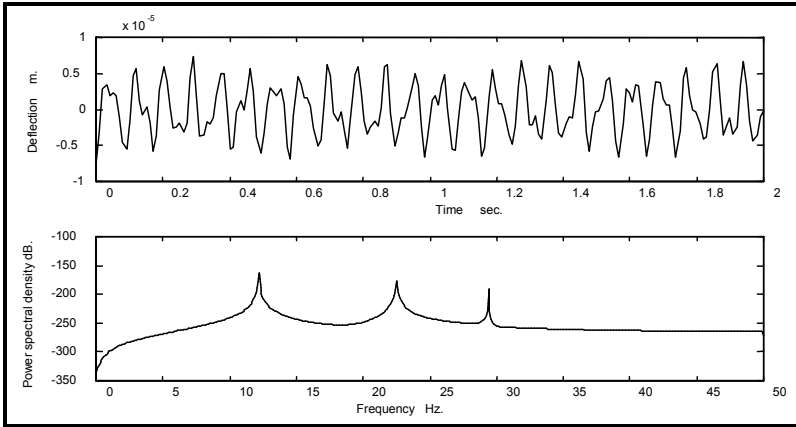


fig2: (a): The response in time and frequency domains at mass 1

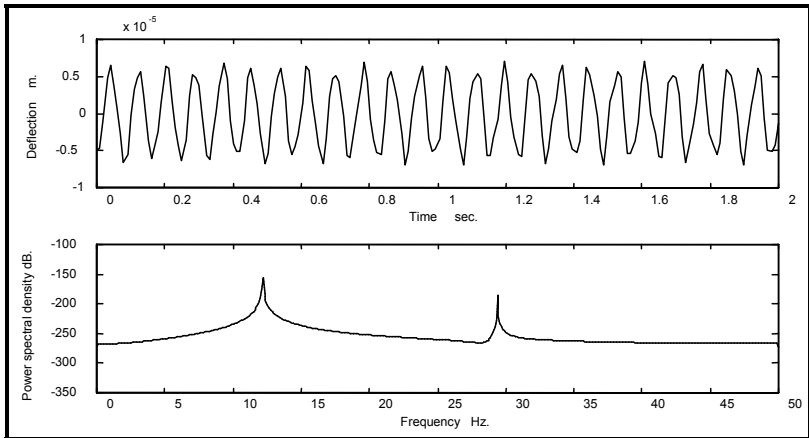


fig2: (b): The response in time and frequency domains at mass 2

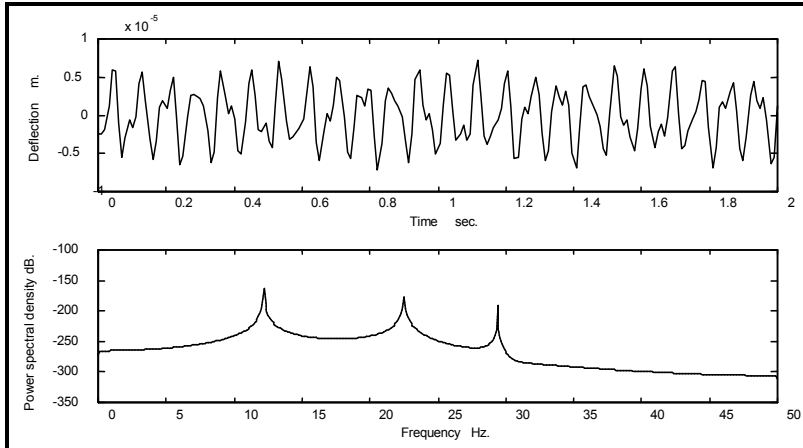
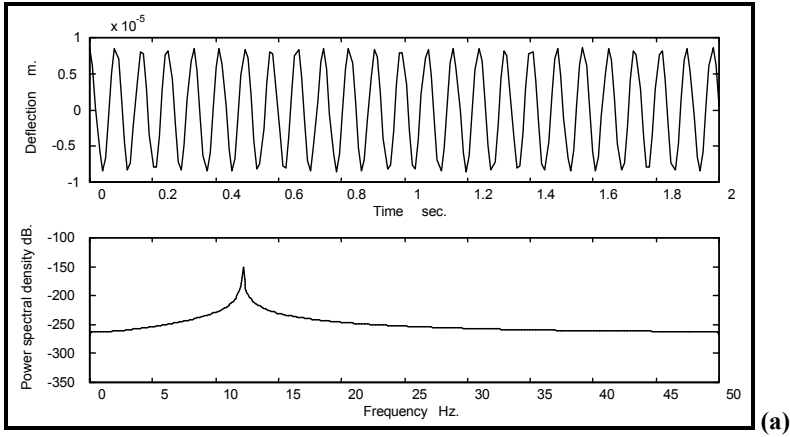


fig2: (c): The response in time and frequency domains at mass 3

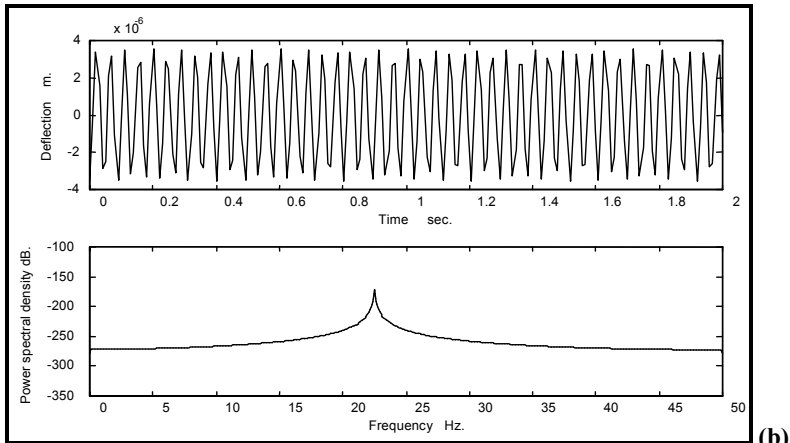
Figure 2: The response of three DOF system

It is obvious that the U matrix is actually the Φ matrix of the system, and the Ψ matrix is then calculated from U as explained previously. The decoupled single DOF outputs are then determined utilizing equation (2.41), and the results are shown in Figure 3. Now, the system natural frequencies can be estimated directly from either the time or frequency of those outputs. The values of these frequencies are,

$$\omega_{n1} = 76.4 \text{ rad/sec} \quad \omega_{n2} = 142.91 \text{ rad/sec} \quad \omega_{n3} = 184.8 \text{ rad/sec}$$



(a)



(b)

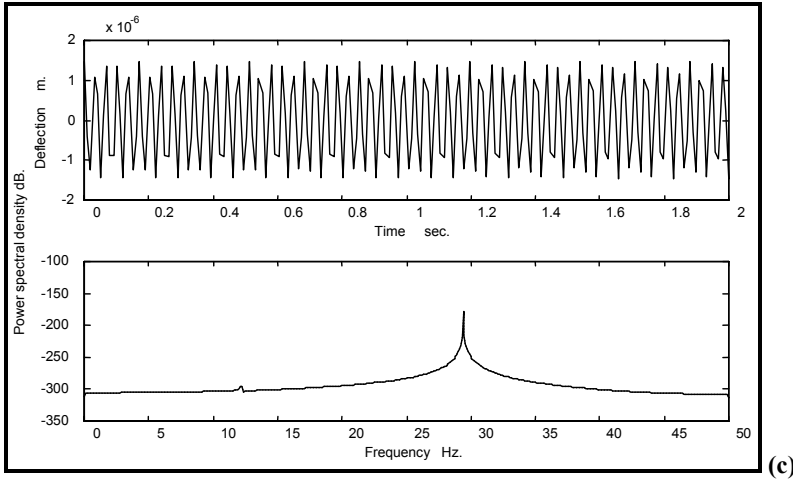


Figure 3: The new de-coupled responses, which represent three single DOF systems

This example shows the approach capability for calculating or estimating accurately its characteristics from theoretical or experimental data. The small discrepancy between the theoretical and estimated natural frequencies is due to the resolution of the recorded data. The estimation can be improved by increasing the number of the sampled data, as well as decreasing the sampling time Δt .

3.2 Five Degrees of Freedom System

A five DOF system with the following mass and stiffness matrices,

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad kg$$

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \times 10^4 \quad \frac{N}{m}$$

is selected, and the orthogonal transformation techniques are implemented to calculate the system parameters, which are found to be,

$$\Omega = \begin{bmatrix} 8.234 & 0 & 0 & 0 & 0 \\ 0 & 15.916 & 0 & 0 & 0 \\ 0 & 0 & 22.51 & 0 & 0 \\ 0 & 0 & 0 & 27.57 & 0 \\ 0 & 0 & 0 & 0 & 30.74 \end{bmatrix} \quad Hz$$

$$\Phi = \begin{bmatrix} 0.29 & 0.58 & 0.58 & 0.58 & 0.29 \\ 0.43 & 0.5 & 0.0 & -0.5 & -0.43 \\ 0.58 & 0.0 & -0.58 & 0.0 & 0.58 \\ 0.43 & -0.5 & 0.0 & 0.5 & -0.43 \\ 0.29 & -0.58 & 0.58 & -0.58 & 0.29 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.74 & 0.5 & 0.0 & -1.0 & -1.74 \\ 2.0 & 0.0 & -1.0 & 0.0 & 2.0 \\ 1.74 & -0.5 & 0.0 & 1.0 & -1.74 \\ 1.0 & -1.0 & 1.0 & -1.0 & 1.0 \end{bmatrix}$$

The output displacements at the masses are generated using the same step input force, sampling time dt , and the number of sampled points as in the first system. However, only the first three natural frequencies of the system are included in the data as illustrated in Figure 4, for the response at mass 1. Again, the HM-QR methods are applied of matrix A, which is prepared from these coupled data, and the resulting U matrix is,

$$U = \begin{bmatrix} 0.289 & 0.5008 & -0.577 & 0.577 & 0.011 \\ 0.5002 & 0.4998 & 0.0012 & -0.687 & 0.169 \\ 0.577 & -0.0012 & 0.5774 & 0.2989 & -0.494 \\ 0.4998 & -0.5002 & -0.0009 & 0.169 & 0.6865 \\ 0.2886 & -0.4992 & -0.5778 & -0.2789 & -0.5058 \end{bmatrix}$$

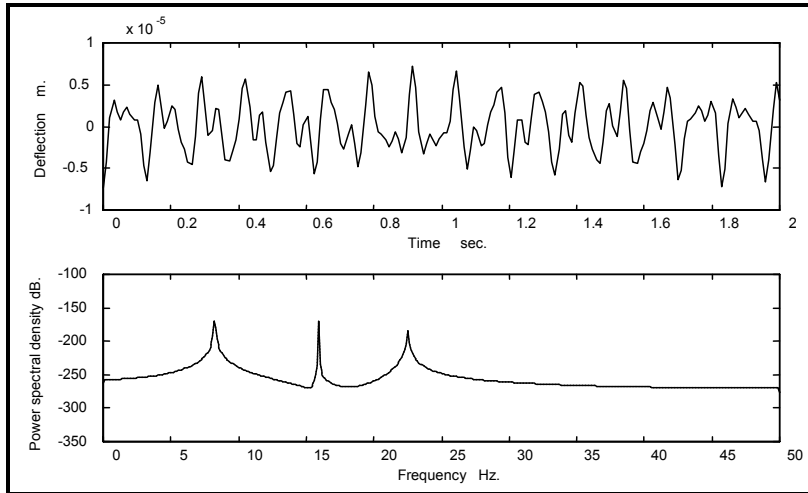
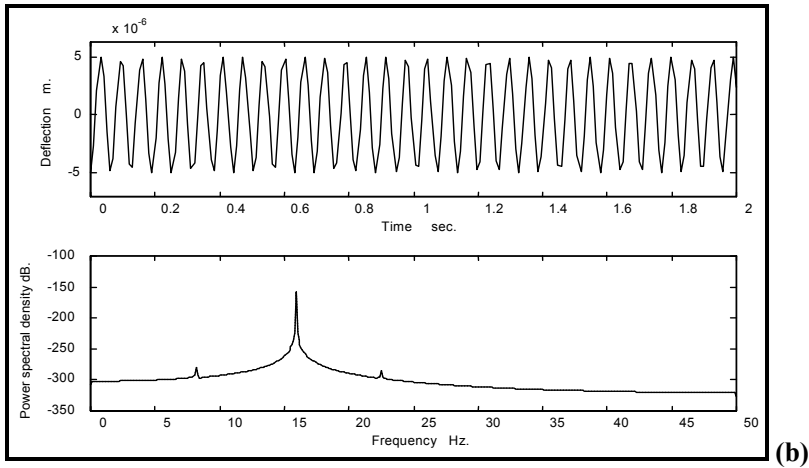
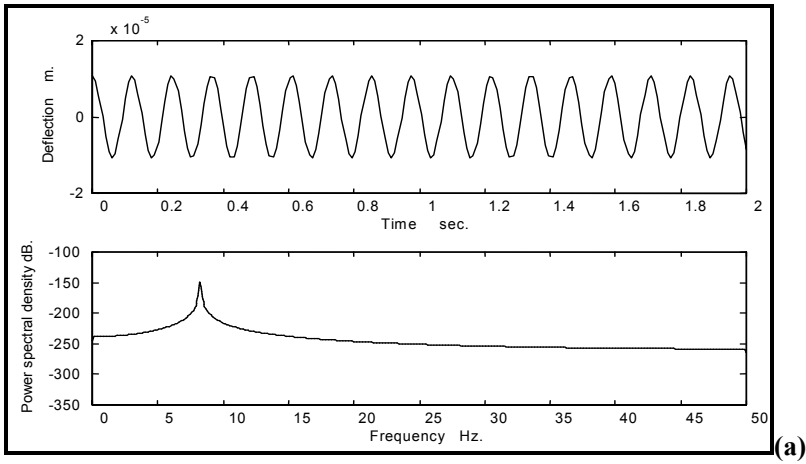


Figure 4: The response in time and frequency domains at mass 1 for five DOF system showing the first three natural frequencies in the recorded data

There is no doubt that the first three columns in matrix U are the same as those of matrix Φ , while the last two represent only the effect of contributions of the last two frequencies on the recorded data. Thus, it is possible to de-couple the original outputs to three single DOF systems using U' matrix for the transformation as shown in Figure 5. The natural frequencies of the system are estimated from these new data, and they were found to be,

$$f_{n1} = 8.23 \text{ Hz} \quad f_{n2} = 16.05 \text{ Hz} \quad f_{n3} = 22.745 \text{ Hz}$$

The method proves its ability to identify the parameters with quite good accuracy, of a system with truncated measured data.



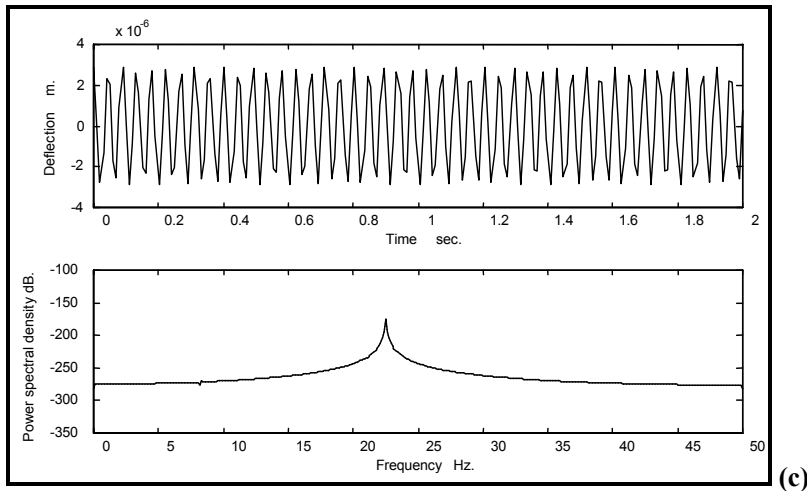


Figure 5: The new de-coupled responses, which represent three single DOF systems

3.3 Cantilever Beam

A beam is fixed at one end, while the other end is free selected as a simple real flexible structure. Its length is 0.8 m, modulus of rigidity (EI) is 0.100105598 Nm^2 , and the mass per unit length is 0.05933274 kg/m .

The beam is modeled as 29 DOF lumped system, which will give only the first three natural frequencies with good accuracy. This is due to the fact of modeling a distributive structure of infinite DOF by a finite DOF model. In the present study, only the first two natural frequencies are considered excited under operational environments as shown in Figure 6.

The orthogonal transformation is applied to the model matrices, i.e. M , and K , as well as to the calculated responses to calculate, and estimate the structure characteristics respectively. The de-coupled two output data demonstrate clearly single DOF systems as illustrated in Figure 7. The calculated and estimated natural frequencies, with their corresponding mode shapes are presented in Tables (1) and (2) respectively. These two tables show the success of the HM-QR method in finding the structure parameters under consideration efficiently.

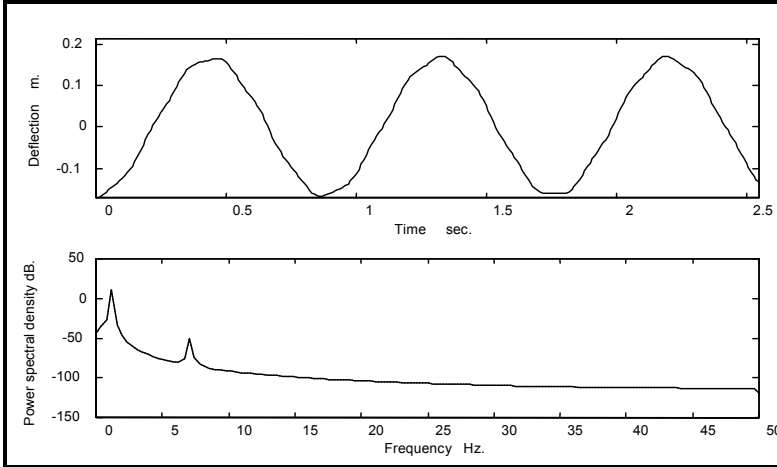


Figure 6: The response in time and frequency domains at the free end of the cantilever beam

Table (1): The calculated and estimated natural frequencies of the cantilever beam

Natural Frequencies rad/sec	
Calculated	Estimated
7.1321	7.39
44.6358	44.352

4. Conclusions

The paper presents the HM-QR method as a technique that can be used to calculate the system natural frequencies, and their corresponding mode shapes from given mass, and stiffness matrices. The same approach can be adopted to find directly the normalized mode shapes from the covariance matrix, which is generated from the system coupled time outputs utilizing the PCA routine. De-couple single DOF data from the original coupled outputs can be determined so that the natural frequency may be estimated directly from this new set of data. The method is

applied to three systems with full or truncated displacement outputs, and it was found that the calculated or estimated system characteristics are accurate enough to use the approach as a successful routine for such identifications.

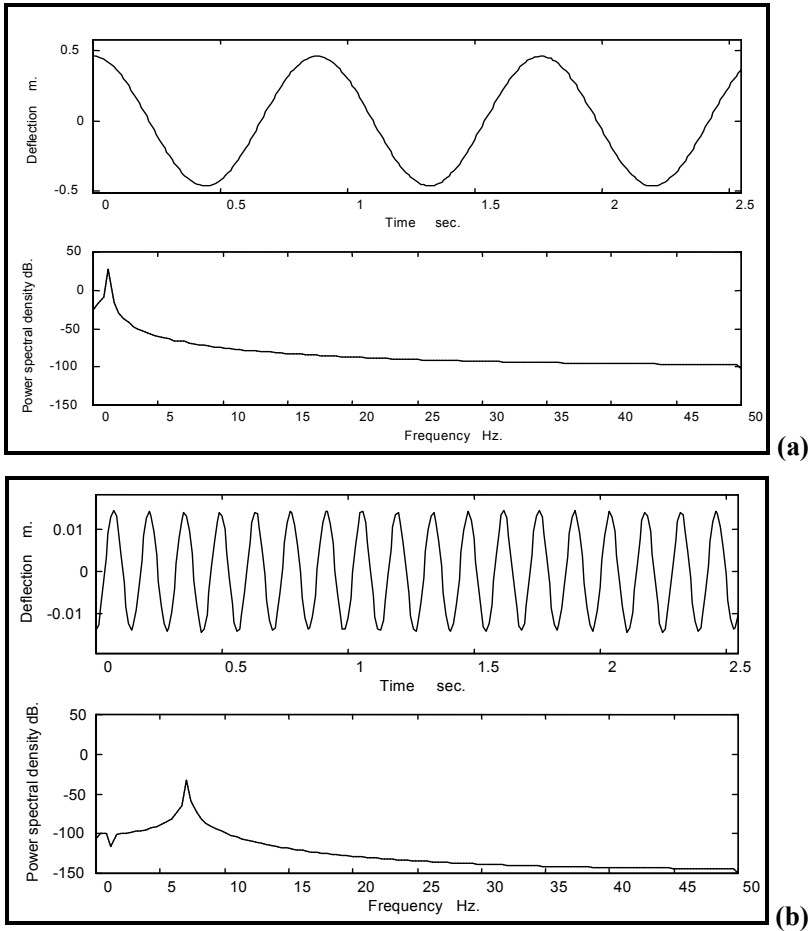


Figure 7: The New de-coupled responses for the first two natural frequencies of the cantilever beam

Table (2)
The calculated and estimated mode shapes of the cantilever beam

Normalized Mode Shapes			
Calculated		Estimated	
1.0	1.0	1.0	1.0
0.953	0.828	0.952	0.826
0.906	0.657	0.905	0.653
0.859	0.486	0.857	0.482
0.812	0.318	0.81	0.313
0.765	0.155	0.763	0.159
0.719	-0.0027	0.716	-0.009
0.672	-0.151	0.669	-0.157
0.626	-0.289	0.622	-0.295
0.581	-0.414	0.577	-0.418
0.535	-0.523	0.531	-0.527
0.491	-0.616	0.487	-0.619
0.447	-0.69	0.443	-0.692
0.405	-0.745	0.40	-0.746
0.363	-0.78	0.359	-0.779
0.323	-0.794	0.319	-0.793
0.284	-0.789	0.28	-0.787
0.247	-0.765	0.243	-0.763
0.212	-0.724	0.209	-0.721
0.179	-0.668	0.176	-0.665
0.147	-0.599	0.145	-0.596
0.119	-0.52	0.116	-0.517
0.093	-0.434	0.091	-0.432
0.069	-0.346	0.068	-0.344
0.049	-0.26	0.048	-0.258
0.032	-0.179	0.031	-0.177
0.018	-0.107	0.018	-0.107
0.0008	-0.051	0.0008	-0.05
0.0002	-0.014	0.0002	-0.013

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