The Effective Length of Columns by Interaction of Bifurcation Analysis and AISC Specification

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Abstract

For a column buckled in the elastic stage the magnitude of the buckling load has a unique value, but in the inelastic stage this magnitude depends on the particular loading path at which buckling occurs.

An iteration procedure is proposed to find the effective length of inelastic column where the flexural rigidity of a compression member is taken as the product of the tangent–modulus E_t and the moment of inertia I. The tangent – modulus at any stress level is obtained from the AISC stress–strain curve.

A computer program is written to find the effective length of columns including the effect of tension member and spread of material yield. A numerical example is presented and validity of the program is verified by checking the output results to be consistent with the AISC specification.

1. Introduction

Two different approaches are commonly used in column analysis. The approach load-deflection approach, which attempts to solve a column problem by tracing its load-deflection behavior throughout the entire range of loading up to ultimate load. The second and simpler approach known as eigenvalue approach attempts to find the maximum strength of a column in a direct manner without calculating the deflection. In this approach, an ideal or perfect column and loading conditions are assumed. The column is assumed to be an ideal one without geometrical or material imperfections. It is loaded in such an ideal manner that in the case of centrically loaded column, the load does not produce transverse deflection until the buckling load, or more accurately, the bifurcation load is reached. In this approach, the determination of the effective length ratio of the column is required. The strength of real column with realistic end support conditions could be determined from that of an equivalent pinned-end one. The length of the latter was equal to the effective length of the former, and elastic as well as inelastic solutions for the stability of the perfectly straight column showed that the distance between the inflection points of the buckled member was equal to KL, the effective length.

For a column buckled in the elastic range, the magnitude of the buckling load has a unique value, but in the inelastic range this magnitude depends on the particular loading path at which buckling occurs. The loading path associated with the tangent-modulus load is such that it is the largest load at which a column can buckle without strain reversal in the cross sections. In the Technical Memorandum No.1 entitled "The Basic Column Formula." (Issue May 19, 1952), the Column Research Council [2] recommended this approach. The following key sentence in this memorandum sums up this:

It is the considered opinion of the Column Research Council that the tangent modulus formula for the buckling strength affords a proper basis for the establishment of working load formula.

Against this background, an iteration procedure is suggested to find the effective length of inelastic column where the flexural rigidity of compression member is taken as the product of the tangent-modulus E_t and the moment of inertia I. The tangent-modulus at any stress level may be obtained from the AISC stress-strain curve or any other assumed curve.

2. Bifurcation Analysis of Rigid Frames

At bifurcation, the critical primary axial forces alone are capable of maintaining equilibrium and compatibility without externally applied forces. In matrix form

$$\{P\} = [K]\,\{x\} = 0 \qquad (1)$$

The trivial solution of Equation (1) means all displacement is zero, which is the primary condition. The nontrivial solution requires that the determinant of the [K] matrix be zero; thus the stability criterion is commonly stated as

Det [K] = 0

Matrix [K] is called the stability stiffness matrix. It is expressed as the sum of a first-order $[K_1]$ matrix and a second-order $[K_2]$ matrix thus,

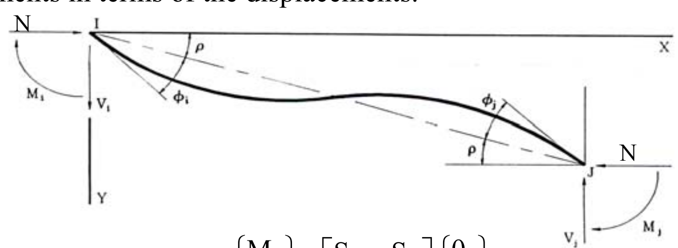
$[K] = [K_1] + [K_2]$

in which $[K_1] = [ASA^T]$

Where:

[A] is the static matrix that expresses the external joint force in terms of the forces in the elements.

[S] is the element stiffness matrix that expresses the forces in the elements in terms of the displacements.



$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{bmatrix} \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

Fig - 1 - Element Forces

If no primary force is applied on the element the element stiffness matrix is given as

$$S_{11} = S_{22} = \frac{4EI}{L}$$

$$S_{21} = S_{12} = \frac{2EI}{L}$$

To account for the effect of a primary axial compressive force the secondary moment is included in the differential equation of the elastic curve [3]. The stiffness are decreased and given as

$$S_{11} = S_{22} = \left[\frac{\phi \sin\phi - \phi^2 \cos\phi}{2 - 2\cos\phi - \phi\sin\phi}\right]\frac{EI}{L}$$

$$S_{21} = S_{12} = \left[\frac{\phi^2 - \phi\sin\phi}{2 - 2\cos\phi - \phi\sin\phi}\right]\frac{EI}{L}$$

$$\text{Where } \phi = \sqrt{\frac{N}{EI}}$$

L – length of the element

N – Primary axial force

EI – flexural rigidity for the element

The element stiffness factors accounting for the effect of primary axial tensile force yield the following expression [3]:

$$S_{11} = S_{22} = \left[\frac{\phi^2 \cosh\phi - \phi\sinh\phi}{2 - 2\cosh\phi - \phi\sinh\phi}\right]\frac{EI}{L}$$

$$S_{21} = S_{12} = \left[\frac{\phi\sinh\phi - \phi^2}{2 - 2\cosh\phi - \phi\sinh\phi}\right]\frac{EI}{L}$$

$[A^T]$ is the deformation matrix and it is the transpose of matrix [A]

$[K_2]$ expresses the external sidesway forces which can be balanced by the end shear forces arisen from the secondary moment couples due to sidesway displacement. In large problem, particularly in rigid frames having nonrectangular joint, it may be more convenient to let the computer generate the $[K_2]$ matrix by the formula derived in reference [3]:

$$[K_2]_{NPS\times NPS} = -\frac{EI_C\,\phi_C^2}{L_C^2}[C]_{NPS\times NMWN}\{G\}_{NPS\times NMWN}\left[C^T\right]_{NMWN\times NPS} \quad (3)$$

in which

NPS = degree of freedom in sidesway

NMWN = number of members under primary axial forces

[C] is a matrix expressing the sidesway forces which may be balanced by the end shear forces acting counter clockwise on member ends

[G] is a diagonal matrix with the element on the mth row or column equal to $\frac{\alpha_m}{L_m}$, α_m is load ratio.

2.1. Definitions for the critical standard stability angle $(\phi_{cr})_c$ and the effective length ratio K_m of the mth member [8]

The determinant of the stability stiffness matrix converges to zero for a sequence of critical values of the buckling load factor N_{cr}. The fundamental mode is the buckled condition occurs at the lowest value of N_{cr}, which is the one usually needed in practice. The name buckling factor N_{cr} will only mean that it is the lowest value.

Under axial compression and tension the flexibility and stiffness coefficients of an mth flexural member are function of end angle ϕ_m defined as $\phi_m = L_m\sqrt{\frac{\alpha_m N}{EI_m}}$ the angle ϕ_m may be called the stability angle of the mth member. If some values of length and moment of inertia (which do not necessarily coincide with the actual length or moment of inertia of any member) are chosen as the standard values L_c and I_c, the standard stability angle ϕ_c is, then $\phi_c = L_c\sqrt{\frac{N}{EI_c}}$.

The ratio β_m of ϕ_m to ϕ_c is $\beta_m = \frac{\phi_m}{\phi_c} = \frac{L_m}{L_c}\sqrt{\frac{\alpha_m}{\left(I_m/I_c\right)}}$ thus, for each assigned value of the standard stability angle ϕ_c, there is a value of the determinant of the stability stiffness matrix of the entire structure. The lowest value of ϕ_c at which this determinant is zero is the critical standard stability angle $(\phi_{cr})_c = L_c\sqrt{\frac{N_{cr}}{EI_c}}$.

The effective length $K_m L_m$ of the mth member in a rigid frame has been defined in steel design specification as the equivalent member length whose Euler load is equal $\alpha_m N_{cr}$; thus $\frac{\pi^2 EI_m}{(K_m L_m)^2} = \alpha_m N_{cr}$ from which

$$K_m = \sqrt{\frac{\pi^2 EI_m}{\alpha_m N_{cr} L_m^2}} = \sqrt{\frac{\pi^2}{(\phi_{cr})_m^2}} = \frac{\pi}{(\phi_{cr})_m} = \frac{\pi}{\beta_m (\phi_{cr})_c} \qquad (4)$$

K_m in equation (4) is the effective length ratio of the mth member in a rigid frame.

3. The AISC Column Formula for Compression Member

Numerous experimental tests have been made in order to find the ultimate load of the column. Various mathematical functions have been used in order to fit the curve of test results in reasonably accurate, yet practical manner. Figure (2) shows typical column strength.

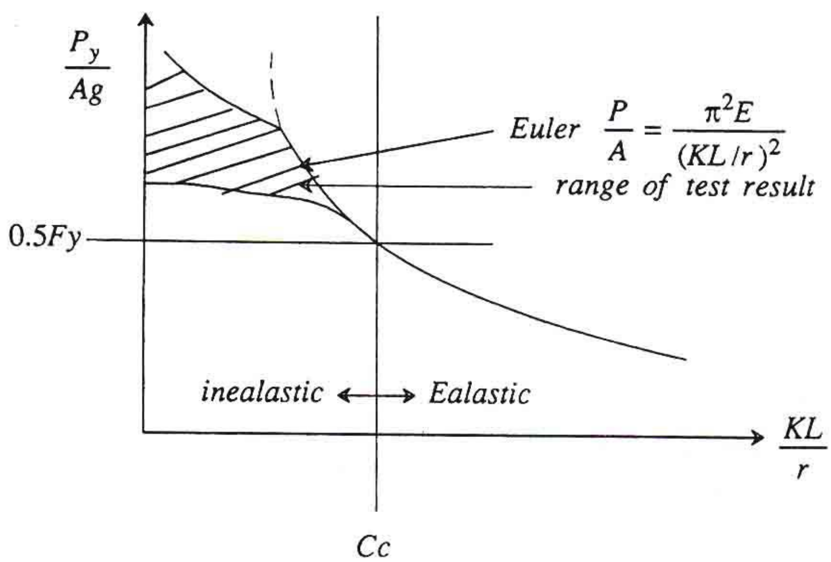


Fig. 2. Typical range of column strength vs slenderness ratios [6]

As a general approach the well known Euler buckling formula is used for the range of elastic buckling

$$F_{cr} = \frac{\pi^2 E}{(KL/r)^2} \qquad (5)$$

This formula is in reasonable agreement with the experimental results, for high slenderness ratio.

The curve-fitting approach is used in inelastic buckling range. The AISC formula for the column strength which is governs the inelastic buckling of the column.

$$F_{cr} = F_y \left[1 - \frac{(KL/r)^2}{2 C_C^2} \right] \qquad (6)$$

Where C_C is the slenderness ratio at column stress equal to 0.5 F_y (Assumed as the proportional limit).

The allowable stress for the column using Euler formula where slenderness ratios exceeding $\frac{KL}{r} = C_C$, elastic buckling

$$F_a = \frac{\pi^2 E}{F.S.(KL/r)^2} \qquad (7)$$

The allowable stress for short column with slenderness ratios less than C_C where inelastic buckling controls.

$$F_a = \frac{F_y}{F.S.}\left[1 - \frac{(KL/r)^2}{2C_C^2}\right] \qquad (8)$$

In order to find F.S. (Factor of safety), transition curve is used [5]. This transition curve can be expressed as a mathematical function of the form:

$$F.S. = \frac{5}{3} + \frac{3}{8}\frac{KL/r}{C_C} - \frac{1}{8}\left(\frac{KL/r}{C_C}\right)^3$$

3.1.Tangent Modulus and Basic Tangent Model Theory

Engesser and Consider [6] were the first to modify Euler equation. Model of elasticity has the potential of being variable. In their theory, column remains straight until failure and the modulus of elasticity at failure is tangent to the stress-strain curve, as shown in Figure 3. Thus, the modified Euler's equation becomes

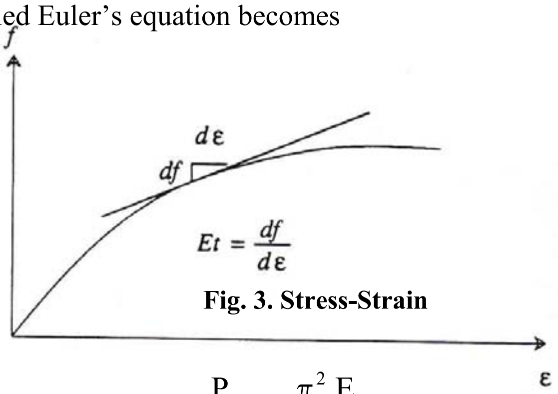


Fig. 3. Stress-Strain

$$F_{cr} = \frac{P_t}{A} = \frac{\pi^2 E_t}{(KL/r)^2} \qquad (9)$$

In elastic buckling where $f_{cr} < 0.5F_y$ or in another expression $\frac{KL}{r} > C_C$.

Equation (9) applies therefore, $E_t = E$.

In inelastic range where $f_{cr} > 0.5\,F_y$ or in another word $\frac{KL}{r} < C_C$.

$$F_{cr} = \frac{\pi^2\,E_t}{(KL/r)^2} \qquad (10)$$

For the same range, AISC formula gives after multiplied by a factor to account for ultimate stress design instead of working stress design

$$F_{cr} = \frac{\frac{23}{12}\left[1 - \frac{(KL/r)^2}{2\,C_C^2}\right]F_y}{\frac{5}{3} + \frac{3}{8}\left(\frac{(KL/r)}{C_C}\right) - \frac{1}{8}\left(\frac{(KL/r)}{C_C}\right)^3} \qquad (11)$$

Substituting for F_{cr} from Equation (10) in Equation (11) gives

$$\frac{\pi^2\,E_t}{(KL/r)^2} = \frac{\frac{23}{12}\left[1 - \frac{(KL/r)^2}{2\,C_C^2}\right]F_y}{\frac{5}{3} + \frac{3}{8}\left(\frac{(KL/r)}{C_C}\right) - \frac{1}{8}\left(\frac{(KL/r)}{C_C}\right)^3} \qquad (12)$$

which gives

$$E_t = \frac{\frac{23}{12}\,(KL/r)^2\left[1 - \frac{(KL/r)^2}{2\,C_C^2}\right]F_y}{\pi^2\left[\frac{5}{3} + \frac{3}{8}\left(\frac{(KL/r)}{C_C}\right) - \frac{1}{8}\left(\frac{(KL/r)}{C_C}\right)^3\right]} \qquad (13)$$

In order to find the upper limit for Equation (13), substitute $\frac{KL}{r} = 0$

$$F_{cr} = \frac{69}{60}F_y$$

Therefore, Equation (13) is valid when

$$\frac{69}{60}F_y \geq F_{cr} \geq 0.5F_y$$

For $F_{cr} > \frac{69}{60}F_y = 1.15F_y$ substitute:

$$1.15F_y = \frac{\pi^2 E_t}{(KL/r)^2}$$

$$E_t = \frac{1.15F_y}{\pi^2}(KL/r)^2$$

4. Strength as Design Criterion for Tension Members

Figure 4. shows stress-strain relation for steel members under tension stress. The design of tension member is simply of providing a member with a sufficient cross-sectional area to resist the applied loads. Stability is only of second concern.

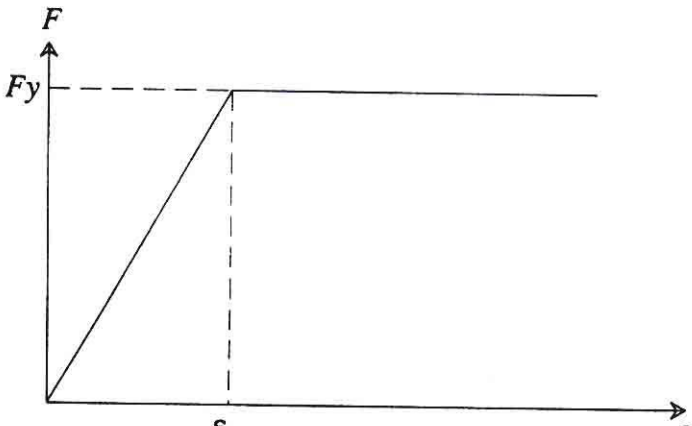


Fig. 4. Stress-Strain

$$f_a = \frac{T}{A_g} \leq 0.6 F_y$$

$$f_a = \frac{T}{A_e} \leq 0.5 F_y$$

Where

f_a – allowable stress

A_g – cross sectional area

A_e – effective net area

4.1. Stiffness as Design Criterion for Tension Members

Although stability is not a criterion in the design of tension members, limiting their length is necessary to prevent the member of becoming too flexible.

L/r limitation:

	AISC	ASHTO
Main members	240	200
Secondary members	300	240
Members subjected to reversal	-	140

Where r is the least radius of gyration and L is length of member.

5. The Computer Program

a computer program is written by this author. A flow-chart of the program is given in figure 5.

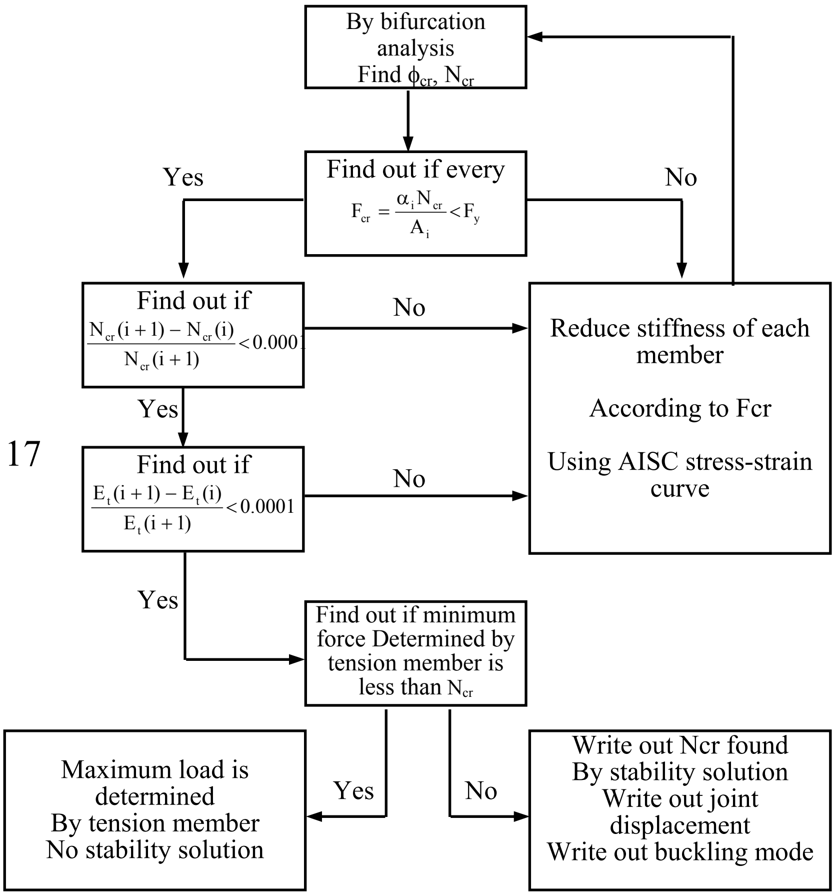


Fig. 5. Flow-Chart

This program takes the structure data as an input. In addition, it takes an arbitrary value of ϕ_c (assumed in this program to be 0.9). After calculating the stability matrix [K] using the starting value of ϕ_c, the determinant of the [K] matrix should be positive or zero. If the determinant was negative, the starting value of ϕ_c should be changed to a lesser value.

The program is capable of performing a bifurcation analysis for a structure, which has members under no primary axial force, members under primary axial compressive force, and members under primary axial tensile force. The difference when analyzing members who are under primary axial force or not occurs in calculating the stiffness coefficients.

When inputting the data for the structure, the ratio of axial load for the member is zero if there is no primary axial force applied to this member. The ratio α is positive or negative if the applied primary axial force is compressive or tensile respectively.

6. Numerical application to a two-bay non-rectangular rigid frame

Figure 6 shows the frame with hinged bases. Steel is A50 Young's modulus is 200 GPa.

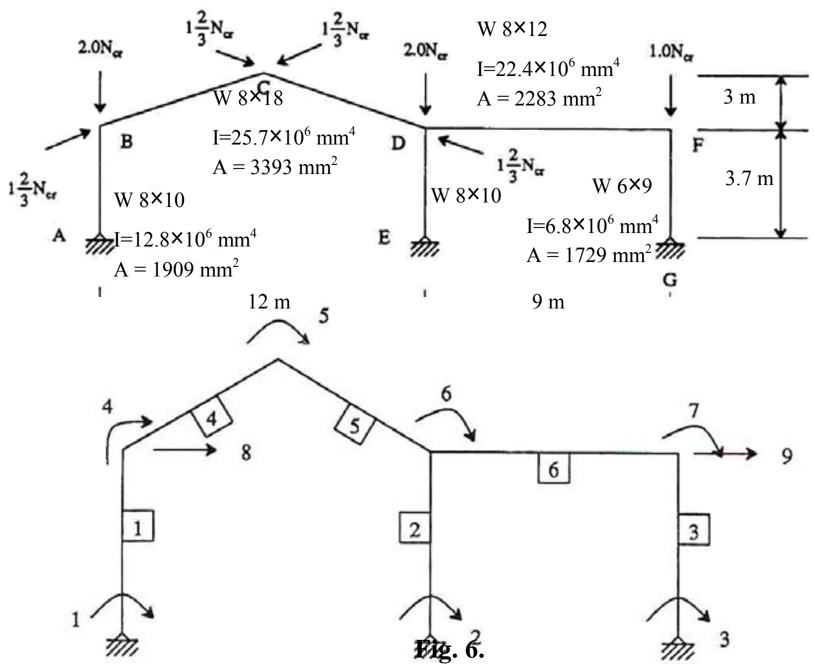


Fig. 6.

The results for the analysis are shown in table 1 taken from the computer output.

In this case $N_{cr} = 371.53$ KN in the first iteration, however, the critical buckling load causes a yielding in column 1 and column 2. Since yielding occurs, reduction in member stiffness has to be done following the procedure explained. Iteration 14 is the last iteration. The results of the first and last iteration are given in table 1.

	element	First iteration	Last iteration
N_{cr}		371.53 KN	294.50 KN
K E_t F_{cr}	AB ED	1.595 140 Gpa 389 Mpa	1.262 99 Gpa 308 Mpa
K E_t F_{cr}	GF	1.646 192 Gpa 214 Mpa	1.894 200 Gpa 170 Mpa
K E_t F_{cr}	BC CD	1.330 199 Gpa 182 Mpa	1.493 200 Gpa 144 Mpa

Table 1

An output check can be made to show that the final results are consistent with the AISC specification.

Column 1, 2 (AB, ED):

$$C_C = 106.9987902$$

$$\frac{KL}{r} = \frac{1.26244819\,(3.7)\,m}{81.94\,mm} = 56.35684815$$

Using Eq (13)

$$E_t = \frac{\frac{23}{12}(KL/r)^2\left[1-\frac{(KL/r)^2}{2C_C^2}\right]F_y}{\pi^2\left[\frac{5}{3}+\frac{3}{8}\left(\frac{(KL/r)}{C_C}\right)-\frac{1}{8}\left(\frac{(KL/r)}{C_C}\right)^3\right]} = 99\,Gpa$$

$E_t = 99\,Gpa$ Check with the output since $\frac{KL}{r}$ is less than C_C.

From the output the critical stress $F_{cr} = 308\,Mpa$. The small discrepancy in the allowable stress is within the tolerance limit set by the criteria for end of iteration procedure.

Column 3 GF:

From the output

$$K = 1.84944355 \Rightarrow \frac{KL}{r} = 107.6586786 > C_C = 106.99$$

Using Eq. (9)

$$\frac{\alpha N_{cr}}{A} = \frac{\pi^2 E_t}{(KL/r)^2}$$

$$E_t = \frac{1.0(294.50\,KN)(KL/r)^2}{\pi^2 (1729\,mm^2)} = 200 \text{ Gpa}$$

$E_t = 200$ Gpa (Check with the output)

Since $\frac{KL}{r}$ is greater than C_C, using the AISC Eq. (7)

$$F_a = \frac{\pi^2 E}{\frac{23}{12}(KL/r)^2} = 88.77\,Mpa$$

Which is the allowable stress in the column. From the output the critical stress is $F_{cr} = 170\,Mpa$ which when multiplied by the safety factor of $\frac{12}{23}$ gives
88.69 Mpa. The small discrepancy in the allowable stress is within the tolerance limit set by the criteria for end of iteration procedure.

Compression members 4,5 (BC, CD):

From the output

$$k = 1.49387133 \Rightarrow \frac{KL}{r} = 116.8503374$$

Using Eq. (9) $E_t = \dfrac{1.666(294.50\,KN)(116.8503374)^2}{(3393\,mm^2)\,\pi^2}$

$E_t = 200\,Gpa$

Which is check with the output

Since $\frac{KL}{r} > C_C$, using Eq. (7)

$$F_a = \frac{12\pi^2 E}{23\left(\frac{KL}{r}\right)^2} = 75.40\,\text{Mpa}$$

Which is the allowable stress in the members BC and CD. From the output the critical stress is $F_{cr} = 144\,\text{Mpa}$. Which is when multiplied by the safety factor of $\frac{12}{23}$ gives 75.13 Mpa. Thus the output is in agreement with the AISC specification.

Table 2. Gives a comparison between results of this analysis with results obtained using direct method from the American steel code [5]

Member	**Effective Length Factor K**		
	Analysis Results		**Direct code results**
	First Iteration	**Last Iteration**	
AB	1.595	1.262	1.84
ED	1.595	1.262	1.65
GF	1.646	1.844	1.80
BC	1.330	1.493	1.37
CD	1.330	1.493	1.30

Table 2

7- conclusion

A procedure is presented to investigate the effect of tension members and spread of material yield on the frame stability.

Stability analysis of frames is performed using the modified element stiffness materices.

The classical procedure of neglecting axial deformation and searching for the loads creating the conditions of bifurcation is applied.

The study confirms the points of the iteration procedure made for the inelastic solution of the buckling problem. The validity of the program is verified by means of comparison between the various applications to the program. The output checks are consistent with the AISC specifications.

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