The Influence Of Reduced Numerical Integration On The Stability of Space-Time Finite Elements

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Abstract

Rectangular space-time finite elements (STFE) applied for axial vibrations of bar were considered using the closed form integration and the numerical integration of the stiffness matrix. These STFE which are conditional stable when we use the closed form integration (or exact numerical integration) become unconditional stable when we use reduced numerical integration (reduced number of points relative to time axis).

For triangular STFE applied to axial vibrations, the use of numerical integration with one point corresponds to exact integration, and has the same conditional stable results of closed form integration.

Reduced numerical integration of rectangular STFE applied to flexural vibrations of beams and to plane stress-strain vibrations has shown unconditional stability.

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Introduction

It is quite known that dynamic analysis is gaining an increasing importance. Tall (or long) and thin structures are sensitive to dynamic effects. Aircraft impact on nuclear reactors (or other important buildings) is now more reality than ever before.
In recent decades the space-time finite elements method introduced itself as an alternative and attractive way of solving dynamic problems. The idea of the method returns back to the general theory of finite elements [١]. Then the method was developed in Poland by the works of [٢, ٣] and others. On the basis of scientific cooperation the method became a research topic at Weimar University, Germany [٤, ٥].

Space-Time Finite Element Method

In the traditional way of solving dynamic problems we use finite elements to discrete structure in space, Fig. ١-a and the following set of equations is obtained.

$$M\ddot{U} + C\dot{U} + KU = F \qquad (١)$$

Where

- M mass matrix
- C damping matrix
- K stiffness matrix
- F force vector
- U displacement vector

Time integration methods (like central difference, Newmark, Wilson, etc.) are then applied to solve this set of equations for U.
The main characteristic of these methods is the stability.
Unstable results are characterized by the unlimited increase of the displacements.
Conditional stable methods require that the time step must be less than (or equal to) the critical time step in order to have stable results ($\Delta t \le \Delta t_{cr}$).

The unconditional stable methods have the advantage that the, time step is not restricted by a critical time step for stability requirements, and only should be chosen due to accuracy requirements.

In the method of space-time finite elements the time axis is introduced to the space dimensions. From the space element and the time step we obtain space-time element (e), and the whole elements build the united space-time element (E), Fig. ١-b.

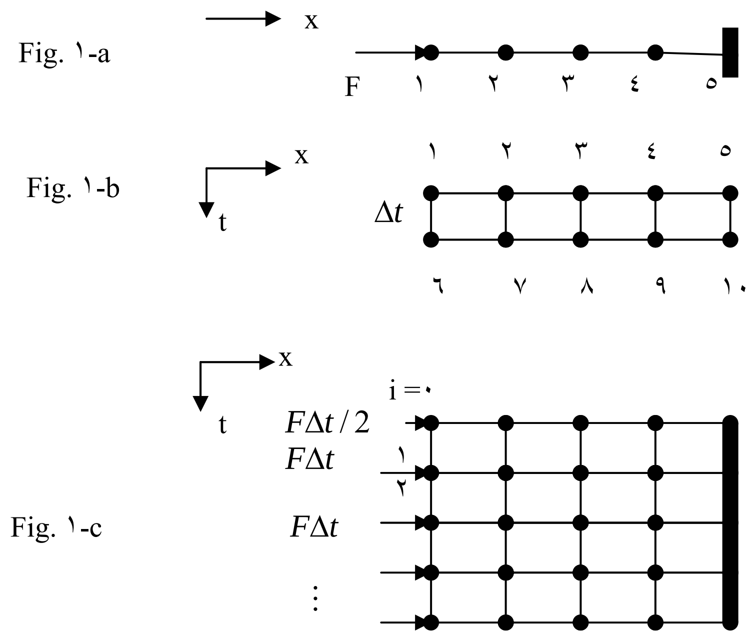


Fig. ١

Considering the successive time steps we get the shape in Fig. ١-c, the force F leads to impulses, and the equilibrium of impulses at these time steps leads to the following system.

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} A & B & & \\ C & D+A & B & \\ & C & D+A & B \\ & & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \end{bmatrix} \qquad (٢\text{-a})$$

Where

$$K_E = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (٢\text{-b})$$

Stiffness matrix of the united space-time

Q Impulse vector

At the start we have

$$Q_0 = AU_0 + BU_1 \qquad (٣\text{-a})$$

Solving for U_1 we get

$$U_1 = B^{-1}(Q_0 - AU_0) \qquad (٣\text{-b})$$

At any subsequent time step we have

$$Q_i = CU_{i-1} + (D+A)U_i + BU_{i+1} \qquad (٤\text{-a})$$

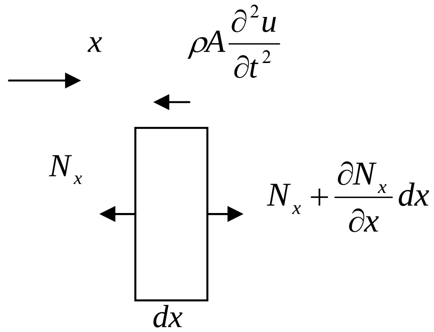
Solving for U_{i+1} leads to

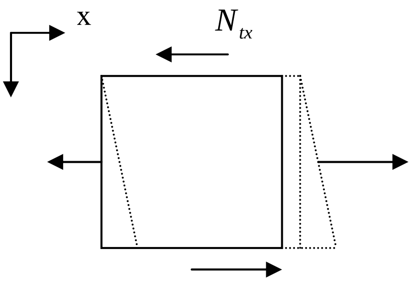
$$U_{i+1} = B^{-1}[Q_i - CU_{i-1} - (D+A)U_i] \qquad (٤\text{-b})$$

Axial Vibrations of Bars

In Fig. ٢-a we consider a differential space element.

Fig. ٢-a





t

dt

Fig. ٢-b N_x $N_x + \frac{\partial N_x}{\partial x} dx$

dx $N_{tx} + \frac{\partial N_{tx}}{\partial t} dt$

Fig. ٢

The dynamic equation of the differential space element (Fig. ٢-a) can be written as follows.

$$\frac{\partial N_x}{\partial x} - \rho A \frac{\partial^2 u}{\partial t^2} = 0 \qquad (٥\text{-a})$$

$$N_x = EA \frac{\partial u}{\partial x} \qquad (٥\text{-b})$$

$$EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} = 0 \qquad (٥\text{-c})$$

Introducing the following symbols.

$$N_{tx} = -\rho A \frac{\partial u}{\partial t} \qquad (٦)$$

The equilibrium of forces (impulses) acting on the differential space-time (Fig. ٢-b) will be identical with the dynamic equation.

$$\frac{\partial N_x}{\partial x} dxdt + \frac{\partial N_{tx}}{\partial t} dxdt = 0 \qquad (٧)$$

For the axial vibration of bar we can write

$$\begin{bmatrix} N_x \\ N_{tx} \end{bmatrix} = \begin{bmatrix} EA & \\ & -\rho A \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{bmatrix} \cdot u(x,t) \qquad (٨)$$

As general terms we have

$$[\sigma] = \begin{bmatrix} N_x \\ N_{tx} \end{bmatrix}$$

$$[E] = \begin{bmatrix} EA & \\ & -\rho A \end{bmatrix} \qquad (٩)$$

$$[\varepsilon] = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial t} \end{bmatrix} \cdot u(x,t)$$

After we define the differential element we can go forward to consider the finite element, (Fig. ٣).

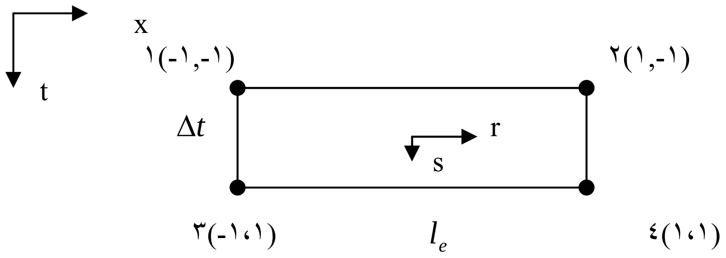


Fig. ٣-a

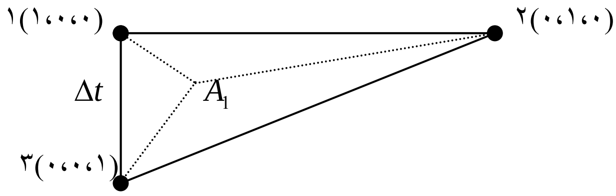


Fig. ٣-b

Fig. ٣

Introducing the shape functions N_i we can calculate the stiffness matrix according to the following formula.

$$K_{ij} = \int \left[EA \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} - \rho A \frac{\partial N_i}{\partial t} \frac{\partial N_j}{\partial t} \right] dxdt \qquad (١٠)$$

Rectangular STFE

For rectangular STFE (Fig. ٣-a) we can write.

$$u(x,t) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad (١١\text{-a})$$

Shape functions can take the following form

$$\begin{aligned} N_1 &= \frac{1}{4}(1-r)(1-s) \\ N_2 &= \frac{1}{4}(1+r)(1-s) \\ N_3 &= \frac{1}{4}(1-r)(1+s) \\ N_4 &= \frac{1}{4}(1+r)(1+s) \end{aligned} \qquad (١١\text{-b})$$

Where r, s the local coordinates

The stiffness matrix gets the following form according to performing the integrals in closed form.

$$K = \frac{EA\Delta t}{6l_e} \begin{bmatrix} 2-2\lambda & -2-\lambda & 1+2\lambda & -1+\lambda \\ -2-\lambda & 2-2\lambda & -1+\lambda & 1+2\lambda \\ 1+2\lambda & -1+\lambda & 2-2\lambda & -2-\lambda \\ -1+\lambda & 1+2\lambda & -2-\lambda & 2-2\lambda \end{bmatrix} \qquad (١٢\text{-a})$$

With

$$\lambda = \frac{l_e^2}{c^2 \Delta t^2}, \; c^2 = \frac{E}{\rho} \qquad (١٢\text{-b})$$

Triangular STFE

For triangular STFE we can write.

$$u(x,t) = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad (١٣\text{-a})$$

Shape functions can be written in the following form.

$$N_1 = r = \frac{A_1}{A} = a_1 x + b_1 t + c_1 \qquad (١٣\text{-b})$$

$$N_2 = s = \frac{A_2}{A} = a_2 x + b_2 t + c_2$$

$$N_3 = 1 - r - s = \frac{A_3}{A} = a_3 x + b_3 t + c_3$$

Where

$$\frac{A_i}{A} = \frac{\det \begin{vmatrix} x & t & 1 \\ x_{i+1} & t_{i+1} & 1 \\ x_{i+2} & t_{i+2} & 1 \end{vmatrix}}{\det \begin{vmatrix} x_1 & t_1 & 1 \\ x_2 & t_2 & 1 \\ x_3 & t_3 & 1 \end{vmatrix}} \qquad (١٣\text{-c})$$

Area coordinates, the determinants are positive according to the shown numbering (Fig. ٣-b).

Performing the integrals in closed form lead to the following stiffness matrix.

$$K_e = \frac{EA\Delta t}{2l_e} \begin{bmatrix} 1-\lambda & -1 & \lambda \\ -1 & 1 & \\ \lambda & & -\lambda \end{bmatrix} \qquad (١٤)$$

Numerical Integration

The stiffness matrix can be numerically calculated by using Gauss points. It takes the following form.

$$K_{ij} = \sum_{I=1}^{n} W\left[EA\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} - \rho A\frac{\partial Ni}{\partial t}\frac{\partial N_j}{\partial t}\right]\det J$$

(١٥)

Where

W Weight of Gauss point

J Jacob matrix

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial s} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial t}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial t}{\partial s} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial t} \end{bmatrix} = J \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial t} \end{bmatrix} \qquad (١٦)$$

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial t} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial s} \end{bmatrix} \qquad (١٧)$$

The Jacob matrix can be calculated as follows.

$$\frac{\partial x}{\partial r} = \Sigma\frac{\partial N_i}{\partial r}x_i \qquad \frac{\partial t}{\partial r} = \Sigma\frac{\partial N_i}{\partial r}t_i \qquad (١٨\text{-a})$$

$$\frac{\partial x}{\partial s} = \Sigma\frac{\partial N_i}{\partial s}x_i \qquad \frac{\partial t}{\partial s} = \Sigma\frac{\partial N_i}{\partial s}t_i \qquad (١٨\text{-b})$$

Numerical integration was performed using Gauss points.

For rectangular STFE the use of four Gauss points corresponds to the exact integration, the use of two (or one) Gauss point presents a reduced integration.

For triangular STFE the use of one integration point corresponds to the exact integration.

Example ١

The example (Fig. ٤-a) was considered with one STFE [٥], (Fig. ٤-b).

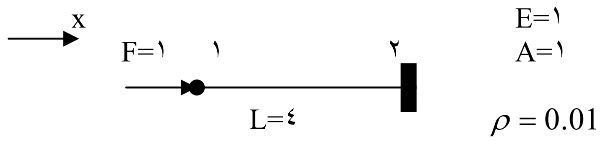


Fig. ٤-a

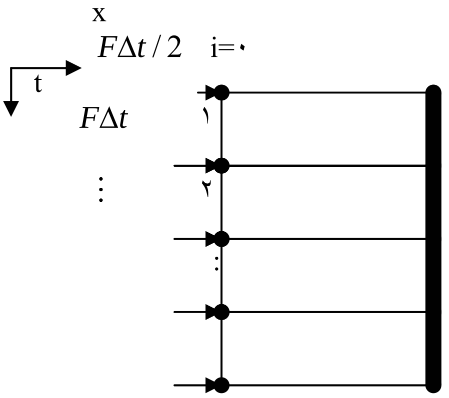
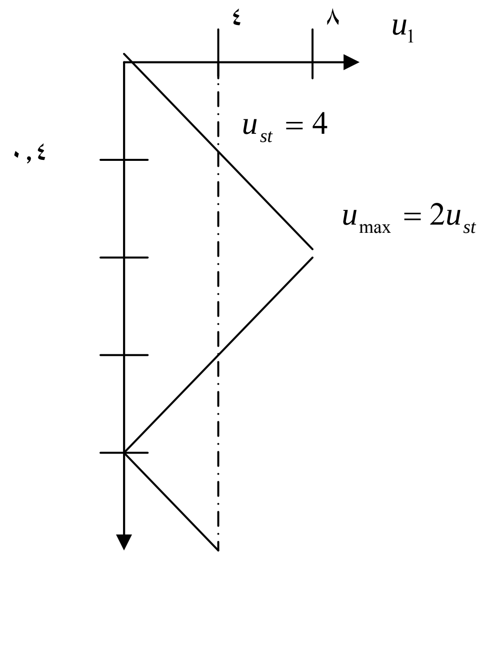


Fig. ٤-b



١٦

٠,٨

Fig. ٤-c

١,٢

١,٦

Fig. ٤

t

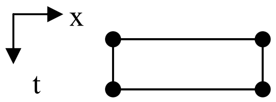
<u>Case ١-a</u>: Rectangular STFE, closed form integration

Using the closed form of integration of the stiffness matrix of the STFE with four nodes (١٢-a), we get the results (Fig. ٤-c and Table ١-a) for the axial displacements of free end with time and for different time steps.

It is proven that the results are unstable when ($\Delta t > 0.8$); i.e. the STFE in this case is conditionally stable.

Table ١: Axial vibrations (Fig. ٤)

Table ١-a



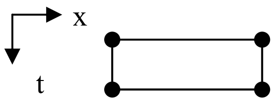
$l_e = 4$

Closed form integration, or
Exact numerical integration;
(٤ Gauss points)

	٠,٤	٠,٨	٠,٨١	(unstable)
١	٤	٨	٨,٠٦٥٩٧٦	
٢	٨	٠	-٠,٦٦٠	
٣	٤	٨	٨,٦٠٧	
٤	٠	٠	-١,٠٩٩٧	
٥	٤	٨	٩,٧٦	
٦	٨	٠	-٢,٦١١	

٧	٤	٨	١١,٦٨	

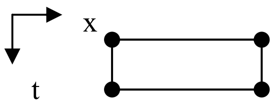
Table ١-b:



Reduced numerical integration;
(٢ Gauss points)

	٠,٤	٠,٨	١	١٠	١٠٠
١	٣,٤٢٨٥٧٢	٦	٦,٥٩٣٤	٧,٩٨٣٠	٧,٩٩٩٨
٢	٧,٨٣٦٧	٦	٤,٦٣٧١	٠,٠٦٨٠	٠,٠٠٠٦٨٣
٣	٥,٦٦٧٦	٠	٠,٥٨٠٤	٧,٨٤٧٦	٧,٩٩٨٥
٤	٠,٦٣٩٧	٦	٧,٧٩٧٠	٠,٢٦٩٦	٠,٠٠٢٧٣٣
٥	١,٣٧٢٣	٦	٢,٤٩٥٩	٧,٥٨١٥	٧,٩٩٥٧
٦	٦,٦٠٩٥	٠	٢,١٥٣٣	٠,٥٩٨٠	٠,٠٠٦١٥
٧	٧,٣٧٣٣	٦	٧,٨٩٨٧	٧,١٩٣٦	٧,٩٩١٦

Table ١-c:

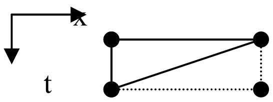


Reduced numerical integration;
(١ Gauss points)

	٠,٤	٠,٨	١	١٠	١٠٠
١	٤	٦,٤	٦,٨٩٦٦	٧,٩٨٧٢	٧,٩٩٩٩
٢	٨	٥,١٢	٣,٨٠٥٠	٠,٠٥١٠٣	٠,٠٠٠٥١٣

٣	٤	٠,٢٥٦	١,٣٨٥٩	٧,٨٨٥٥	٧,٩٩٨٨
٤	٠	٧,٣٧٢٨	٧,٩٨١٠	٠,٢٠٢٨	٠,٠٠٢٠٥
٥	٤	٣,٦٩٦٦	٠,٨٤٨٦	٧,٦٨٤٦	٧,٩٩٦٨
٦	٨	٠,٩٩١٢	٤,٥٨٣١	٠,٤٥١٥	٠,٠٠٤٦١
٧	٤	٧,٩١٣٩	٦,٣٠٦٩	٧,٣٨٩٧	٧,٩٩٣٧

Table ١-d:



Closed form integration, or
Exact numerical integration;
(١ Gauss points)

	٠,٤	٠,٥٦٦	٠,٦	(unstable)
١	٤	٨	٩,٠٠٠٠	
٢	٨	٠	-٤,٥٠٠٠	
٣	٤	٨	٢٠,٢٥٠	
٤	٠	٠	-٢٨,١٢٥	
٥	٤	٨	٦٨,٠٦٣	
٦	٨	٠	-١٢٤,٠٣	
٧	٤	٨	٢٦٠,٠٢	

<u>Case ١-b,c</u>: Rectangular STFE, numerical integration

Calculating the stiffness matrix by using four Gauss-points (exact integration) we get the same results as in case ١-a, (Table ١-a).
The use of two (or one) Gauss points presents reduced integration (Table ١-b, c).

As an essential remark, the reduction of Gauss points was made relative to time axis.
As can be seen from the results, we find that the reduced integration has made the rectangular STFE unconditional stable.

Case ١-d: Triangular STFE, closed form and numerical integration

Because numerical integration with one point corresponds to the closed form integration we get identical results that are conditional stable (Table ١-d).

Other Tests

STFE (Fig. ٥-a) with reduced numerical integration were tested on flexural vibrations of beams in [٥], and they had shown unconditional stability.

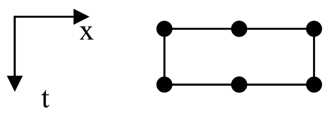


Fig. ٥-a (٣ or ٢) Gauss points

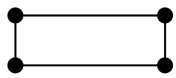


Fig. ٥-b (٢ or ١) Gauss points

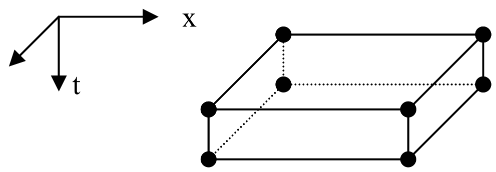


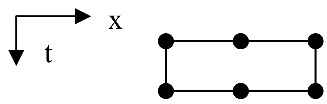
Fig. ٥-c ٤ Gauss points

Fig. ٥

We can now extend the characteristics of these elements to the axial vibrations of bars, (Table ٢-a, b).

Table ٢: Axial vibrations (Fig. ٤)

Table ٢-a:

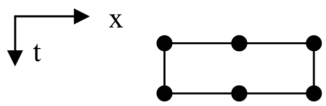


٣ Gauss points

$l_e = 4$

	٠,٤	٠,٨	١٠	
١	٣,٨١٨٢	٦,٠٨٦٥	٧,٩٨٣٠	
٢	٦,٧٤٣٨	٥,٥٠٤٤	٠,٠٦٧٩	
٣	٥,٧٨٣٦	١,٢١٥٣	٧,٨٤٧٨	
٤	٢,٧٦٠٥	٤,٤٨٨٠	٠,٢٦٨٩	
٥	٠,٢٧٠٧	٦,٥٠١٣	٧,٥٨٣٠	
٦	٤,٩٥٦٣	١,٥١١٦	٠,٥٩٥٠	
٧	٧,٢٧١٧	٣,٢٣٩١	٧,١٩٩١	

Table ٢-b:



٢ Gauss points

	٠,٤	٠,٨	١٠		
١	٣,٨٤٠٠	٦,٠٨٧٨	٧,٩٨٣٠		
٢	٦,٦٠٤٨	٥,٤٩١٦	٠,٠٦٧٩		
٣	٦,١٣٧٩	١,٢٧٣٠	٧,٨٤٧٨		
٤	٢,٣٤٢٦	٤,٣٢٦٦	٠,٢٦٨٩		
٥	٠,٣٨١٩	٦,٨١٦٠	٧,٥٨٣٠		
٦	٥,٢٧٨٨	١,٠٦٥٣	٠,٥٩٥٠		
٧	٦,٦٨٨٥	٣,٧٠١٧	٧,١٩٩١		

We can also extend the characteristic of the STFE in Fig. ٥-b, the same element applied for axial vibrations of bars (Case ١-b, c of Example ١), to the flexural vibrations of beams (Table ٣-a, b).

Table ٣: Flexural vibrations

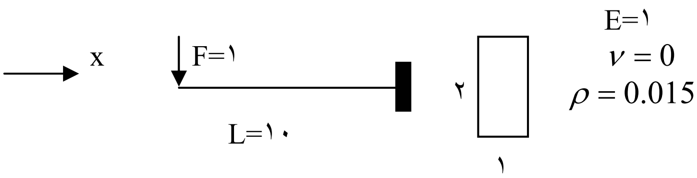
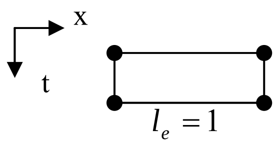


Table ٣-a:

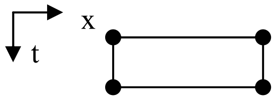


٢ Gauss points

	٥	١٠	١٠٠
١	١٦٢,٨٨	٤٠٤,٤٢	٩١٥,٠٢
٢	٤٧٧,٠٩	٨٧٧,٨٤	٤٨,٧٨٩
٣	٧٨٩,٤٤	٦٨١,٥٢	٨٢٠,١٢
٤	٩٢٢,٥٥	١٠٦,٦٨	١٨٤,٥٢

٥	٧٤٥,٤٨	١٣٢,٠٧	٦٥١,٠١
٦	٤١٧,٠٥	٧٢١,٠٩	٣٩٧,٧٥
٧	١١٩,٧٩	٨٦١,٩٥	٤٤٤,٣٢

Table ٣-b:

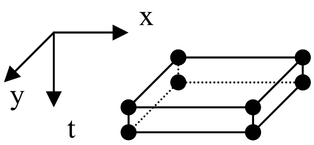


١ Gauss points

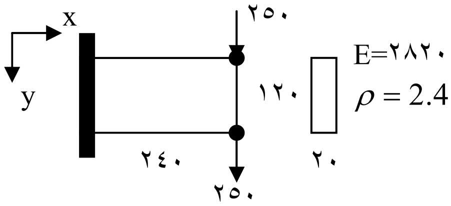
	٥	١٠	١٠٠
١	١٦٧,٧٨	٤٢٣,٤١	١٠٠٦,٦
٢	٤٩٣,٩٧	٩٥١,٠٤	٥٨,٧٣
٣	٨٣٥,٢١	٨٠٦,٩٧	٨٩٢,٧١
٤	١٠١٦,٢٤	١٧٨,٤٣	٢٢٠,٨٨
٥	٨٧٩,١٧	٧٨,٣٥	٦٩٢,٠٦
٦	٥٥٣,٨٣	٦٧٤,٨٢	٤٤٧,٩٨
٧	٢١١,٧٨	٩٩٠,٤٧	٤٥٢,١٢

The STFE (Fig. ٥-c), applied to stress-strain vibrations, has shown unconditional stable results (Table ٤).

Table ٤: Plane stress-strain vibrations



٤ Gauss point



	٥	١٠	١٠٠
١	٠,٠١٢٥٣٣	٠,٠٤٣١٨٥	٠,٢٣١٥
٢	٠,٠٤٧٢٣٢	٠,١٤١٥٩٦	٠,٠٤٠٩٠٥
٣	٠,٠٩٦٩٨٢	٠,٢٢٥٥٤٢	٠,١٥٦٩٤٩
٤	٠,١٥١٩٠٢	٠,٢٣٥٢٤٠	٠,١٣٥٩٧٢
٥	٠,٢٠٠٣٩٠	٠,١٦٣٦٢٦	٠,٠٥٨١٩٩
٦	٠,٢٣٢٥١١	٠,٠٦١٩٩٥	٠,٢٢٠٩٤٦

٧	٠,٢٤٢٠٠	٠,٠٠٢١٨١	٠,٠٠٢٠٢٣
٨	٠,٢٢٦٥٦٠	٠,٠٢٧١٢٣	٠,٢٣٨٣٩٣

Conclusion

The example results of the axial vibrations of bar demonstrated that the use of reduced numerical integration (reduced number of Gauss points relative to time axis) of the stiffness matrix of the rectangular STFE made these STFE unconditional stable, these same STFE are conditional stable when we use the closed form integration (or exact numerical integration); i.e. the stability of rectangular STFE depends on the way we determine the stiffness matrix.

For triangular STFE we get exact numerical integration with one point and the results are conditional stable as the closed form integration.

Reduced numerical integration of rectangular STFE employed for flexural vibrations of beams and plane stress-strain vibrations has shown unconditional stable results.

We can also conclude that the characteristics of the rectangular STFE with reduced integration are not dependent on the application case (for example; axial vibration or flexural vibration); so we can expect that STFE applied to stress-strain case will be unconditional stable when applied to plate bending.

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