[illegible]

السؤال الأول:

$$\frac{dM_Y(t)}{dt} = \nu(1-2t)^{-\frac{\nu}{2}-1} \quad , \quad \frac{d^2M_Y(t)}{dt^2} = \nu(\nu+\ )(1-2t)^{-\frac{\nu}{2}-2}-1$$

$$E(X) = \left.\frac{dM_Y(t)}{dt}\right|_{t=0} = \nu \quad , \quad E(X^2) = \left.\frac{d^2M_Y(t)}{dt}\right|_{t=0} = \nu^2+2\nu$$ 5

$$V(X) = E(X^2) - (EX)^2 = 2\nu$$ 1

ب- 8 بالاستفادة من أ- يمكننا أن نكتب:

$$E\left[\frac{(n-1)S^2}{\sigma^2}\right] = (n-1) \Rightarrow \frac{n-1}{\sigma^2}ES^2 = (n-1) \Rightarrow ES^2 = \sigma^2$$ 4

$$V\left[\frac{(n-1)}{\sigma^2}S^2\right] = 2(n-1) \Rightarrow \frac{(n-1)^2}{\sigma^4}V(S^2) = 2(n-1)$$

$$\Rightarrow V(S^2) = \frac{2\sigma^4}{n-1}$$ 4

٢- 8

$$S_1^2 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n+2} = \frac{n-1}{n+2}S^2 = \frac{n+2-3}{n+2}S^2 = \left(1-\frac{3}{n+2}\right)S^2$$

$$\Rightarrow ES_1^2 = \left(1-\frac{3}{n+2}\right)\sigma^2 = \sigma^2 - \frac{3}{n+2}\sigma^2$$ 4

إذن S_1^2 مقدر متحيز بالنقصان ومقدار التحيز $\frac{3}{n+2}\sigma^2$.

$$V(S_1^2) = V\left[\left(1-\frac{3}{n+2}\right)S^2\right] = \left(1-\frac{3}{n+2}\right)^2 V(S^2)$$ 4

$$= \left(1-\frac{3}{n+2}\right)^2 \cdot \frac{2\sigma^4}{n-1} = \frac{2(n-1)}{(n+2)^2}\sigma^4$$

٣- 4

$$E(X_1-4X_2) = EX_1 - 4EX_2 = \mu - 4\mu = -3\mu$$ 2

$$V(2X_1-3X_2) = 4V(X_1) + 9V(X_2) = 4\sigma^2 + 9\sigma^2 = 13\sigma^2$$ 2

السؤال الثاني:

أ)

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\theta)^2} \right) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

$$\ln L(\theta) = -n \ln\sqrt{2\pi} - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n}(x_i-\theta) = \sum_{i=1}^{n} x_i - n\theta$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0 \Rightarrow \theta = \bar{x}$$

وبما أن $\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -n < 0$ فإن $\boxed{\hat{\theta} = \bar{X}}$

ب) 6

$$\ln f(x,\theta) = -\ln\sqrt{2\pi} - \frac{1}{2}(x-\theta)^2 \Rightarrow \frac{\partial}{\partial\theta}\ln f(x,\theta) = x-\theta$$

$$\Rightarrow \frac{\partial^2}{\partial\theta^2}\ln f(x,\theta) = -1 \Rightarrow I(\theta) = E\left(-\frac{\partial^2 \ln f(X,\theta)}{\partial\theta^2}\right) = 1$$

إذن: $\boxed{I_n(\theta) = n}$

ت) يمكن إثبات ذلك بطريقتين مختلفتين:

الطريقة الأولى:

$$V(\bar{X}) = \frac{\sigma^2}{n} = \frac{1}{n} = \frac{1}{nI(\theta)}$$

الطريقة الثانية:

$$\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\ln f(x_i,\theta) = \sum_{i=1}^{n}(X_i-\theta) = n(\bar{X}-\theta)$$

ج) 4

$$f(x_1, x_2, \ldots, x_n, \theta) = L(\theta) = \left[\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}x_i^2}\right] e^{\theta\sum_{i=1}^{n}x_i - \frac{n}{2}\theta^2}$$

$$= h(x_1, \ldots, x_n)\, g\left(\sum_{i=1}^{n}x_i, \theta\right)$$

وبالتالي يكون حسب قاعدة عاملين الإحصاء $\sum_{i=1}^{n} X_i$ كافٍ لـ θ وبالتالي $\bar{X}$ إحصاء كافٍ لـ θ

د) 4

من الواضح أن: $E(\bar{X}) = \theta$ وأن $V(\bar{X}) = \frac{\sigma^2}{n} = \frac{1}{n}$

وبالتالي $\lim_{n\to\infty} V(\bar{X}) = 0$

إذن $E(\bar{X}) = \theta$ و $\lim_{n\to\infty} V(\bar{X}) = 0$

وبالتالي $\bar{X}$ مقدر متسق لـ θ.

السؤال الثالث:

$$E(S_1) = E\left(\frac{T_1+T_2+T_3}{3}\right) = \frac{1}{3}(3\sigma^2) = \sigma^2$$

$$E(S_2) = E\left(\frac{2T_1+T_2+2T_3}{5}\right) = \frac{1}{5}(5\sigma^2) = \sigma^2 \qquad 3$$

$$E(S_3) = E\left(\frac{T_1+2T_2+T_3}{4}\right) = \frac{1}{4}(4\sigma^2) = \sigma^2$$

أي أن كلاً من S_1 و S_2 و S_2 مقدرات غير متحازة للوسيط θ.

ولدينا:

$$V(S_1) = V\left(\frac{T_1+T_2+T_3}{3}\right) = \frac{1}{9}\left[V(T_1)+V(T_2)+V(T_3)\right] = \frac{1}{9}(2\sigma^2+\sigma^2+2\sigma^2) = \frac{5}{9}\sigma^2 \qquad 3$$

$$V(S_2) = V\left(\frac{2T_1+T_2+2T_3}{5}\right) = \frac{1}{25}\left[4V(T_1)+V(T_2)+4V(T_3)\right] = \frac{1}{25}(8\sigma^2+\sigma^2+8\sigma^2) = \frac{17}{25}\sigma^2 \qquad 3$$

$$V(S_3) = V\left(\frac{T_1+2T_2+T_3}{4}\right) = \frac{1}{16}\left[V(T_1)+4V(T_2)+V(T_3)\right] = \frac{1}{16}(2\sigma^2+4\sigma^2+2\sigma^2) = \frac{8}{16}\sigma^2 = \frac{1}{2}\sigma^2 \qquad 3$$

بالمقارنة نجد أن S_3 هو المقدر الأفضل من بين المقدرات الثلاثة المذكورة، وذلك لأن:

$$V(S_3) \le V(S_1) \le V(S_2) \qquad 3$$

ب) 1)

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^1 \theta x^\theta\,dx = \theta\left[\frac{x^{\theta+1}}{\theta+1}\right]_0^1 = \frac{\theta}{\theta+1} \qquad 3$$

2)

$$\mu = M_1 = \bar{X} \iff \frac{\theta}{\theta+1} = \bar{X} \implies \theta - \theta\bar{X} = \bar{X}$$

$$\implies \theta(1-\bar{X}) = \bar{X} \implies \theta = \frac{\bar{X}}{1-\bar{X}} \qquad 5$$

إذن مقدر العزوم لـ θ:

$$\boxed{\hat{\theta} = \frac{\bar{X}}{1-\bar{X}}}$$

$$\hat{\mu} = \frac{\hat{\theta}}{1+\hat{\theta}} = \frac{\frac{\bar{X}}{1-\bar{X}}}{1+\frac{\bar{X}}{1-\bar{X}}} = \bar{X} \qquad 2$$