سلم تصحيح تحليل (٢) للسنة الثانية رياضيات ف ٢ .

السؤال الأول (٣٠ درجة) $\begin{matrix} 1/15 \\ 2/15 \end{matrix}$

1/ $\lim_{x\to x_0} g(x) = B \iff$

$\forall \varepsilon>0,\ \exists \delta>0,\ 0<\|x-x_0\|<\delta,\ |g(x)-B|<\varepsilon$

نريد اثبات $\lim_{x\to x_0}\left(\frac{1}{g(x)}\right)=\frac{1}{B}$ أي نريد اثبات

$\forall \varepsilon' >0,\ \exists \delta>0,\ 0<\|x-x_0\|<\delta,\ \left|\frac{1}{g(x)}-\frac{1}{B}\right|<\varepsilon'$

$\left|\frac{1}{g(x)}-\frac{1}{B}\right| = \frac{|B-g(x)|}{|B\cdot g(x)|} < \frac{\varepsilon}{|B|\,|g(x)|}\quad (*)$ 3

$|B| = |B-g(x)+g(x)| < |B-g(x)|+|g(x)| < \varepsilon+|g(x)|$

$\Rightarrow |B| < \varepsilon+|g(x)| \Rightarrow |g(x)| > |B|-\varepsilon$ 3

$\Rightarrow \left|\frac{1}{g(x)}\right| < \frac{1}{|B|-\varepsilon}$ 3

و بما أن ε اختيارية نختاره $\varepsilon=\frac{|B|}{2}$ و بالتالي ينتج

$\left|\frac{1}{g(x)}-\frac{1}{B}\right| < \left(\frac{\varepsilon}{|B|}\right)\frac{1}{|B|-\varepsilon} = \frac{\frac{|B|}{2}}{|B|\left(|B|-\frac{|B|}{2}\right)} = \frac{1}{|B|} = \varepsilon'$ 3

$\Rightarrow \lim_{x\to x_0}\frac{1}{g(x)} = \frac{1}{B}$ و هو المطلوب

2/ بما أن f تفاضلية في x_0 فإنه يوجد $\delta>0$ بحيث
إذا كان $\|h\|<\delta$ فإنه يوجد $A\in\mathbb{R}^n$ بحيث

$f(x_0+h)-f(x_0) = \sum_{i=1}^{n} A_i h_i + \eta\|h\|$ 3

حيث $\lim \eta = 0$

$\Rightarrow |f(x_0+h)-f(x_0)| = \left|\sum_{i=1}^{n} A_i h_i + \eta\|h\|\right| \le$ 3
$\sum_{i=1}^{n}|A_i||h_i|+|\eta|\,\|h\|$

$$\Rightarrow |f(c+h)-f(c)| \le \sum_{i=1}^{n} |A_i| \, \|h\| + |\eta| \, \|h\|$$

لأن $\|h_i\| \le \|h\|$ 3

$$\Rightarrow |f(c+h)-f(c)| \le \left\{ \sum_{i=1}^{n} |A_i| + |\eta| \right\} \|h\|$$

وبما أن $\eta \to 0$ عندما 3

$\forall \varepsilon > 0, \ \varepsilon = 1, \ \exists \delta_2 > 0, \ 0 < \|h\| < \delta_2 \ (\|h\| \to 0), \ |\eta - 0| < 1$

$$\Rightarrow |f(c+h)-f(c)| \le \left\{ \sum_{i=1}^{n} |A_i| + 1 \right\} \|h\| \quad (A)$$

ومنه $\exists \delta = \min(\delta_1, \delta_2)$ وبالتالي عندما 3

$\|h\| < \delta$

لنفرض $\sum_{i=1}^{n} |A_i| + 1 = K$ و $c+h = x$

نعوض في (A)

$$|f(x) - f(c)| \le K \, \|x - c\|$$

السؤال الثاني (70 درجة) 20 20 20 10

1) $f(x,y) = e^x \sin y$ $\quad c = (0, \frac{\pi}{2})$

$$f(a+h, b+k) - f(a,b) = \sum_{1}^{\infty} \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(a,b)$$ 3

$f_x(x,y) = e^x \sin y$ $\quad f_x(0, \frac{\pi}{2}) = 1$ $\quad f(0, \frac{\pi}{2}) = 1$

$f_{xx}(x,y) = e^x \sin y$ $\quad f_{xx}(0, \frac{\pi}{2}) = 1$

$f_y(x,y) = e^x \cos y$ $\quad f_y(0, \frac{\pi}{2}) = 0$

$f_{yy}(x,y) = -e^x \sin y$ $\quad f_{yy}(0, \frac{\pi}{2}) = -1$

$f_{xy} = e^x \cos y$ $\quad f_{xy}(0, \frac{\pi}{2}) = 0$

2

$f(\cancel{0}+h, \frac{\pi}{2}+k)-1 = h + \frac{h^2-k^2}{2}$

$e^{h}\sin(\frac{\pi}{2}+k) = 1+h+\frac{h^2-k^2}{2}$ 2

$e^{x}\sin y = 1+x+\frac{x^2}{2}-\frac{(y-\frac{\pi}{2})^2}{2}$ 2

2/ $\forall \varepsilon>0,\ \exists \delta>0,\ \|(x,y)-(0,0)\|<\delta$ 3
$|f(x,y)-f(0,0)|<\varepsilon$

$|f(x,y)-f(0,0)| = \left|\frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}-0\right| < \frac{(x^2+y^2)(x^2+y^2)}{(x^2+y^2)^{\frac{3}{2}}}$

$|f(x,y)-f(0,0)| < \sqrt{x^2+y^2} < \delta$ 3

$\forall \varepsilon>0,\ \exists\ \delta=\varepsilon>0,\ \|(x,y)-(0,0)\|<\delta,\ |f(x,y)-f(0,0)|<\varepsilon$
f متصلة في (0,0) 2

$f(a+h, b+k)-f(a,b) = hf_x(a,b)+kf_y(a,b)+\eta\sqrt{h^2+k^2}$ 3

$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$ 3

$f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$ 3

$\frac{h^2k^2}{(h^2+k^2)^{3/2}} = \sqrt{h^2+k^2}\ \eta$ 3

$\eta = \frac{h^2k^2}{(h^2+k^2)^2}$

نأخذ $h=k$ 3

$\eta = \frac{k^4}{(2k^2)^2} = \frac{k^4}{4k^4} \xrightarrow[k\to 0]{} \frac{1}{4} \neq 0$

3

الدالة غير قابلة للاشتقاق في (0,0) 3

3) $f_x(x,y) = 2y^2 + 2x = 0$ $\left.\begin{matrix} -4y^3 + 4xy = 0 \\ 8y - 4xy = 0 \end{matrix}\right\}$

$f_y(x,y) = 8y - 4xy = 0$

6

$-4y^3 + 8y = 0 \Rightarrow 4y(-y^2+2) = 0 \Rightarrow y = 0 \Rightarrow x = 0$

$y = -\sqrt{2} \Rightarrow x = 2$

$y = +\sqrt{2} \Rightarrow x = 2$

$(0,0)\ (2,\sqrt{2})\ (2,-\sqrt{2})$ وينتج نقاط حرجة

$f_{xx} = 2$

3

$f_{xy} = -4y$

$f_{yy} = 8 - 4x$

من أجل $(0,0)$

$\Delta_1 = f_{xx} = 2 > 0$

$\Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 8 \end{vmatrix} = 16 > 0 \Rightarrow (0,0)$ صغرى محلية

2

$(2,+\sqrt{2})$

$\Delta_1 = 2 > 0,\ \Delta_2 = \begin{vmatrix} 2 & -4\sqrt{2} \\ -4\sqrt{2} & 0 \end{vmatrix} = -32 < 0$

2

نقطة سرجية

$(2,-\sqrt{2})$

2 $\Delta_1 = 2 > 0,\ \Delta_2 = \begin{vmatrix} 2 & 4\sqrt{2} \\ 4\sqrt{2} & 0 \end{vmatrix} = -32 < 0$

نقطة سرجية

---

4) $\|u\| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$\frac{\partial f}{\partial u}(1,0) = \lim_{h \to 0} \frac{f(c+hu) - f(c)}{h}$ 3

$c + hu = (1,0) + \left(-\frac{\sqrt{3}}{2}h, \frac{1}{2}h\right) = \left(1 + \frac{\sqrt{3}}{2}h, \frac{1}{2}h\right)$

$f(c+hu) = 2\left(1 + \frac{\sqrt{3}}{2}h\right)^3 - 3\frac{h^2}{4}$ 3

$= 2\left(1 + \frac{3(3)h^2}{4} + \frac{\sqrt{3}}{2}h + \frac{3\sqrt{3}}{8}h^3\right) - \frac{3}{4}h^2$

$= 2 + \frac{15h^2}{4} + \frac{2\sqrt{3}}{2}h + \frac{6\sqrt{3}}{8}h^3$

4

$$f(c) = f(1,0) = 2$$

$$\frac{\partial f}{\partial u}(1,0) = \lim_{h \to 0} \frac{2 + \frac{15}{4}h^2 + \frac{2\sqrt{3}}{2}h + \frac{6}{8}\sqrt{3}h^3 - 2}{h}$$

$$= \lim_{h \to 0} \frac{15}{4}h + \frac{2\sqrt{3}}{2} + \frac{6}{8}\sqrt{3}h^2 = \sqrt{3}$$